A spin-shift relation of the Ruijsenaars-Schneider Hamiltonian of type C_n

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Abstract

We construct a spin-shift relation of the trigonometric Ruijsenaars-Schneider Hamiltonian of type C_n. This is a successive study of the previous papar [8].

1 Introduction

Ruijsenaars-Schneider systems describe one-dimensional n-particle system with pairwise interaction. In [2], the authors derived the Hamiltonian for the trigonometric one-particle Ruijsenaars- Schneider system from the Gervais-Neveu-Felder equation. We would like to recall their argument in brief. The Cervais-Neveu-Felder equation has its L-invariant form:

$$R_{12}(xq^{-H_3/2})L_{13}(x)L_{23}(x) = L_{23}(x)L_{13}(x)R_{12}(xq^{H_3/2}) \tag{1}$$

with a subscript 3 denote the quantum space. Let us represent $L_{13}(x)$ in the tensor product of the representation $\rho^{(\frac{1}{2})} \otimes \rho^{(j)}$, where $\rho^{(j)}$ is the spin-j representation of $U_q(sl_2)$. Taking the trace on the first space, and still restricting this operator to the space of zero-weight vectors, one obtains

$$H_j = q^{-x\frac{\partial}{\partial x}} + q^{-x\frac{\partial}{\partial x}}\left(1 - \frac{(q^j - q^{-j})(q^{j+1} - q^{-j-1})}{(x - x^{-1})(q^{-1}x - qx^{-1})}\right) \tag{2}$$

when j is an integer. This is the trigonometric one-particle Ruijsenaars- Schneider Hamiltonian. That is the reason we call j the spin of the model insted of the coupling constant of the model. The integrability of the system generated by the Hamiltonian obtained here was solved by using the existence of a spin-shift operator D_j which satisfies

$$H_j D_j = D_j H_{j-1}. \tag{3}$$

In recent work [8], we constructed the spin-shift operator for the trigonometric multi-particle Ruijsenaars-Schneider system in terms of the Dunkl-Cherednik operators.

The purpose of the present paper is to find results analogous to those of [8] for the root system C_n. As a consequence we obtain the spin-shift operator in the language of Weyl group.

This Letter is arranged as follows. In Section 2, we review the basic facts of the Dunkl-Cherednik operators and the Ruijsenaars- Schneider Hamiltonian. Following the similar process as in [8] leads us to obtain the spin-shift relation (Proposition 2) and the explicit form of the spin-shift operator for trigonometric Ruijsenaars- Schneider Hamiltonian of type C_n (Theorem 1) in Section 3.

2 Dunkl-Cherednik operators

In this section we define the Dunkl-Cherednik operators, which are the key tool for our paper. For details, see references ([4]-[6]).

Let V be an n-dimensional real vector space with the basis $\{\epsilon_1, \ldots, \epsilon_n\}$, and a positive definite symmetric bilinear form $(\cdot,\cdot) : V \times V \to \mathbf{R}$ defined by $(\epsilon_i, \epsilon_j) = \delta_{ij}$. Let $R = \{\pm\epsilon_i \pm \epsilon_j \, (1 \le i < j \le n), \, \pm 2\epsilon_i \, (1 \le i \le n)\}$ be the root system of type C_n, $R^+ = \{\epsilon_i \pm \epsilon_j \, (1 \le i < j \le n), \, 2\epsilon_i \, (1 \le i \le n)\}$ the set of positive roots, $\Pi = \{\alpha_i = \epsilon_i - \epsilon_{i+1} \, (1 \le i \le n-1), \alpha_n = 2\epsilon_n\}$ the set of simple roots. For every root $\alpha \in R$, define the coroot $\alpha^\vee := 2\alpha/(\alpha,\alpha)$. Denote by $P = \{\lambda \in V | (\lambda, \alpha_i^\vee) \in \mathbf{Z}\}$ the weight lattice. It has a natural basis of fundamental weights ω_i determined by $(\alpha_j^\vee, \omega_i) = \delta_{ji}$. In similar way we define coweight lattice $P^\vee = \{\lambda^\vee \in V | (\lambda^\vee, \alpha_i) \in \mathbf{Z}\}$ and fundamental coweight $\omega_i^\vee$ determined by $(\alpha_j, \omega_i^\vee) = \delta_{ji}$. As usual we define the highest root $\theta \in R$ by $\theta - \alpha \in \oplus \mathbf{Z}_+ \alpha_i$ for all $\alpha \in R$.

Let $\Lambda_n = \mathbf{Q}(q)[X^{\epsilon_1}, \ldots, X^{\epsilon_n}]$ be a polynomial ring. For every $\alpha \in R$ denote s_α for the reflection with respect to α determined by $s_\alpha(v) = v - (v,\alpha)\alpha^\vee$. The Weyl group $W = W_{C_n} = \langle s_1, \ldots, s_{n-1}, s_n \rangle$ acts on Λ_n as $w.X^\alpha = X^{w(\alpha)}$.

The set of affine roots is $\widetilde{R} = \{\alpha + m\delta | \alpha \in R, m \in \mathbf{Z}\}$, where δ denotes the constant function 1 on V. The simple roots are $a_0 = -\theta + \delta$ and $a_i = \alpha_i \in R$. We use the same symbols $s_i \, (0 \le i \le n)$ to represent the generator for the corresponding affine Weyl group $\widetilde{W} = \langle s_0, s_1, \ldots, s_n \rangle$. We note that $s_0 = \tau(\theta^\vee) s_\theta$, where $\tau(\xi^\vee)$ is defined by $\tau(\xi^\vee) X^\mu = q^{2(\xi^\vee,\mu)} X^\mu$. We define the length $\ell(\lambda^\vee)$ of $\lambda^\vee \in P^\vee$ as the length l of the reduced expression

$$\tau(\lambda^\vee) = s_{i_1} \cdots s_{i_l}. \tag{4}$$

We write $\lambda^\vee \prec \mu\vee$ if $\ell(\lambda^\vee) < \ell(\mu^\vee)$. Suppose for every $\alpha \in \widetilde{R}$ we have a variable t_α such that $t_\alpha = t_{w(\alpha)}$ for every $w \in \widetilde{W}$. Let $\mathbf{Q}_t = \mathbf{Q}(t_\alpha)$ be the field of rational functions in t_α. We now introduce the operators

$$T_i = t_{\alpha_i} s_i + (t_{\alpha_i} - {t_{\alpha_i}}^{-1}) \frac{1}{X^{-\alpha_i} - 1}(s_i - 1) \quad (i = 0, \dots, n). \tag{5}$$

They satisfy the following relations

$$(T_i - t_{\alpha_i})(T_i + {t_{\alpha_i}}^{-1}) = 0 \quad (i = 0, \dots, n), \tag{6}$$
$$T_i T_j = T_j T_i \quad (|i - j| \geq 2), \tag{7}$$
$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad (i = 1, \dots, n-2), \tag{8}$$
$$T_i T_{i+1} T_i T_{i+1} = T_{i+1} T_i T_{i+1} T_i \quad (i = 0, n-1). \tag{9}$$

This means that they realize the representation of the affine Hecke algebra $H(\widetilde{W})$ of $\widetilde{W}$ on $\Lambda_{n,t} = \mathbf{Q}_t(q)[X^{\epsilon_1}, \dots, X^{\epsilon_n}]$.

The Dunkl-Cherednik operator $Y^{\omega_i^\vee}$ $(i = 1, \dots, n)$ are now defined in terms of the operators T_i $(i = 0, \dots, n)$ by the expression

$$Y^{\omega_i^\vee} := (T_0 T_1 \cdots T_{n-1} T_n T_{n-1} \cdots T_i)^i. \tag{10}$$

Using the relations (7)-(9), one can show that the Dunkl-Cherednik operators $Y^{\omega_1^\vee}, \dots, Y^{\omega_n^\vee}$ commute with each other. Moreover they satisfy the following commutation relations with T_i.

$$\begin{cases} T_i Y^{\omega^\vee} - Y^{s_i(\omega^\vee)} T_i = (t_{\alpha_i} - t_{\alpha_i}^{-1}) Y^{\omega^\vee}, & \text{if } (\omega^\vee, \alpha_i) = 1, \\ T_i Y^{\omega^\vee} = Y^{\omega^\vee} T_i, & \text{otherwise.} \end{cases} \tag{11}$$

Set $\hat{R} = \{\alpha \in R \,|\, (\alpha, \omega_n^\vee) = 1\}$. Let $M(q, t_\alpha)$ denote the Macdonald difference operator

$$M(q, t_\alpha) = \sum_{w \in W} \prod_{\alpha \in \hat{R}} \frac{t_\alpha X^{w(\alpha)} - {t_\alpha}^{-1}}{X^{w(\alpha)} - 1} \tau(w(\omega_n^\vee)) \tag{12}$$

It is known that the Macdonald difference operators are obtained in terms of the Dunkl-Cherednik operators.

Proposition 1 *[6]*

$$\left. \sum_{w \in W} Y^{w(\omega_n^\vee)} \right|_{\Lambda_{n,t}^W} = M(q, t_\alpha) \tag{13}$$

Note that $O|_{\Lambda_{n,t}^W}$ means that the action of the operator O is restricted to the W-invariant space $\Lambda_{n,t}^W$.

We will define the Ruijsenaars-Schneider Hamiltonian of type C_n by conjugating the Macdonald difference operator (Equation (12)) by a weight function as follows.

For this purpose we introduce the notation

$$(x;q^2)_l = \prod_{i=0}^{l-1}(1-xq^{2i}). \tag{14}$$

Note that in terms of this notation the Macdonald difference operator with $t_\alpha = q^{l_\alpha}$ can be written as

$$M(q,q^{l_\alpha}) = \sum_{w\in W}\prod_{\alpha\in R^+}\frac{(X^\alpha;q^2)_{l_\alpha}}{(X^{w(\alpha)};q^2)_{l_\alpha}}\tau(w(\omega_n^\vee)) \tag{15}$$

Let

$$\Delta^+ := \prod_{\alpha\in R^+}(X^\alpha;q^2)_{l_\alpha} \tag{16}$$

be the weight function we mentioned above. Consider an operator

$$H_{(l_\alpha)} := \Delta^+ M(q,q^{l_\alpha})(\Delta^+)^{-1}, \tag{17}$$

where $(l_\alpha) = (l_1,l_2) \in \mathbf{Z}_{\geq 0}\times\mathbf{Z}_{\geq 0}$. The nonnegative integers l_1 and l_2 correspond to long root and short root respectively. Now let

$$\begin{aligned}\hat{R}_w^+ &= w(\hat{R})\cap R^+,\\ \hat{R}_w^- &= -w(\hat{R})\cap R^+,\end{aligned}$$

then for $\alpha\in R^+$

$$\tau(w(\omega_n^\vee))(X^\alpha;q^2)_{l_\alpha} = \begin{cases} (q^2X^\alpha;q^2)_{l_\alpha}, & \text{for } \alpha\in\hat{R}_w^+\\ (q^{-2}X^\alpha;q^2)_{l_\alpha}, & \text{for } \alpha\in\hat{R}_w^-\\ (X^\alpha;q^2)_{l_\alpha}, & \text{otherwise.}\end{cases}$$

Therefore the Equation.(17) is

$$H_{(l_\alpha)} = \sum_{w\in W}\prod_{\alpha\in R^+}\frac{q^{2l_\alpha}X^{w(\alpha)}-1}{X^{w(\alpha)}-1}\prod_{\beta\in\hat{R}_w^+}\frac{X^\beta-1}{q^{2l_\beta}X^\beta-1}\prod_{\gamma\in\hat{R}_w^-}\frac{q^{2l_\gamma-2}X^{w(\gamma)}-1}{q^{-2}X^{w(\gamma)}-1}\tau(w(\omega_n^\vee)) \tag{18}$$

We call the operator $H_{(l_\alpha)}$ Ruijsenaars-Schneider Hamiltonian of type C_n and $(l_\alpha)\in\mathbf{Z}_{\geq 0}\times\mathbf{Z}_{\geq 0}$ a spin of the model.

3 Spin-Shift relation

We begin with the equation which follows from the commutativity of the Dunkl-Cherednik operators:

$$\sum_{w\in W}Y^{w(\omega_n^\vee)}\prod_{\alpha\in R^+}\left(t_\alpha^{-1}Y^{\frac{\alpha^\vee}{2}} - t_\alpha Y^{-\frac{\alpha^\vee}{2}}\right) = \prod_{\alpha\in R^+}\left(t_\alpha^{-1}Y^{\frac{\alpha^\vee}{2}} - t_\alpha Y^{-\frac{\alpha^\vee}{2}}\right)\sum_{w\in W}Y^{w(\omega_n^\vee)} \tag{19}$$

Proposition 2 *Let w_0 be the longest element of Weyl group W. Multiplying both sides of Equation (19) by w_0 from the left, and restricting the operators to $\Lambda_{n,t}^W$, we obtain*

$$\begin{aligned}
&\left\{\sum_{w\in W}\prod_{\alpha\in\hat{R}}\frac{t_\alpha X^{w(\alpha)}-t_\alpha^{-1}}{X^{w(\alpha)}-1}\prod_{\beta\in\hat{R}_w^-}\frac{t_\beta X^\beta-t_\beta^{-1}}{t_\beta^{-1}X^\beta-t_\beta}\frac{q^{-2}t_\beta^{-1}X^\beta-t_\beta}{q^{-2}t_\beta X^\beta-t_\beta^{-1}}\tau(w(\omega_n^\vee))\right\}\\
&\times\left\{\prod_{\alpha\in R^+}\frac{t_\alpha X^{w(\alpha)}-t_\alpha^{-1}}{X^\alpha-1}\right.\\
&\quad\left.\times\sum_{w\in W}\mathrm{sgn}(w)w\left(\prod_{i=2}^{\infty}\prod_{\substack{\alpha\in R^+\\ ht(\alpha)\geq i}}\frac{t_\alpha q^{2(i-1)}X^\alpha-t_\alpha^{-1}}{q^{2(i-1)X^\alpha-1}}\tau(\rho^\vee)+\sum_{\lambda^\vee\prec\rho^\vee}G_{\lambda^\vee}\tau(\lambda^\vee)\right)\right\}\\
&=\left\{\prod_{\alpha\in R^+}\frac{t_\alpha X^{w(\alpha)}-t_\alpha^{-1}}{X^\alpha-1}\right.\\
&\quad\left.\times\sum_{w\in W}\mathrm{sgn}(w)w\left(\prod_{i=2}^{\infty}\prod_{\substack{\alpha\in R^+\\ ht(\alpha)\geq i}}\frac{t_\alpha q^{2(i-1)}X^\alpha-t_\alpha^{-1}}{q^{2(i-1)X^\alpha-1}}\tau(\rho^\vee)+\sum_{\lambda^\vee\prec\rho^\vee}G_{\lambda^\vee}\tau(\lambda^\vee)\right)\right\}\\
&\times\left\{\sum_{w\in W}\prod_{\alpha\in\hat{R}}\frac{t_\alpha X^{w(\alpha)}-t_\alpha^{-1}}{X^{w(\alpha)}-1}\tau(w(\omega_n^\vee))\right\}
\end{aligned}\tag{20}$$

where $G_{\lambda^\vee}$ is come meromorphic function in $\Lambda_{n,t}$.

proof. We can prove the proposition in the same manner as in [8]. We will outline the proof.

Since in $\prod_{\alpha\in R^+}(t_\alpha^{-1}Y^{\frac{\alpha^\vee}{2}}-t_\alpha Y^{-\frac{\alpha^\vee}{2}})|_{\Lambda_{n,t}^W}$, the term of $\tau(-\rho^\vee)$ appears only from $Y^{-\rho^\vee}$ one can calculate the coefficient:

$$\prod_{i=1}^{2n-1}\prod_{\substack{\alpha\in R^+\\ ht(\alpha)\geq i}}\frac{t_\alpha-q^{-2(i-1)}t_\alpha^{-1}X^\alpha}{1-q^{-2(i-1)}X^\alpha}.\tag{21}$$

Using the identity

$$\begin{aligned}
&(T_i+t_{\alpha_i}^{-1})\prod_{\alpha\in R^+}(t_\alpha^{-1}Y^{\frac{\alpha^\vee}{2}}-t_\alpha Y^{-\frac{\alpha^\vee}{2}})\\
&\quad=\frac{t_{\alpha_i}^{-1}Y^{-\frac{\alpha_i^\vee}{2}}-t_{\alpha_i}Y^{\frac{\alpha_i^\vee}{2}}}{t_{\alpha_i}^{-1}Y^{\frac{\alpha_i^\vee}{2}}-t_{\alpha_i}Y^{-\frac{\alpha_i^\vee}{2}}}\prod_{\alpha\in R^+}(t_\alpha^{-1}Y^{\frac{\alpha^\vee}{2}}-t_\alpha Y^{-\frac{\alpha^\vee}{2}})(T_i-t_{\alpha_i}),
\end{aligned}$$

which can be shown by a direct calculation, we have

$$s_i\frac{1}{\prod_{\alpha\in R^+}\frac{t_\alpha-t_\alpha^{-1}X^\alpha}{1-X^\alpha}}\prod_{\alpha\in R^+}(t_\alpha^{-1}Y^{\frac{\alpha^\vee}{2}}-t_\alpha Y^{-\frac{\alpha^\vee}{2}})\Bigg|_{\Lambda_{n,t}^W}$$

$$= -\frac{1}{\prod_{\alpha\in R^+}\frac{t_\alpha - t_\alpha^{-1}X^\alpha}{1-X^\alpha}}\prod_{\alpha\in R^+}(t_\alpha^{-1}Y^{\frac{\alpha^\vee}{2}} - t_\alpha Y^{-\frac{\alpha^\vee}{2}})\Big|_{\Lambda_{n,t}^W} \tag{22}$$

Thus this invarinaceness up to (± 1) with Equation (21) leads to

$$\prod_{\alpha\in R^+}(t_\alpha^{-1}Y^{\frac{\alpha^\vee}{2}} - t_\alpha Y^{-\frac{\alpha^\vee}{2}})\Big|_{\Lambda_{n,t}^W} = \prod_{\alpha\in R^+}\frac{t_\alpha - t_\alpha^{-1}X^\alpha}{1-X^\alpha} \tag{23}$$

$$\times\sum_{w\in W}\mathrm{sgn}(w)w\left(\prod_{i=2}^{2n-1}\prod_{\substack{\alpha\in R^+\\ ht(\alpha)\geq i}}\frac{t_\alpha - q^{-2(i-1)}t_\alpha^{-1}X^\alpha}{1-q^{-2(i-1)}X^\alpha}\tau(-\rho^\vee) + \sum_{\lambda^\vee \prec \rho^\vee} G_{\lambda^\vee}\tau(-\lambda^\vee)\right),$$

where $G_{\lambda^\vee}$ is some meromorphic function in $\Lambda_{n,t}$. Since $\sum_{w\in W}Y^{w(\omega_n^\vee)}|_{\Lambda_{n,t}^W}$ is Weyl invariant, the right hand side of Equation (20) is proved.

Next we step to prove the left hand side of Equation (20). Let $\Omega^\vee$ be the Weyl orbit of $\omega_n^\vee$. Because of the invarianceness up to (± 1) (Equation (22)), we find that the equation

$$w_0\sum_{w\in W}Y^{w(\omega_n^\vee)}\prod_{\alpha\in R^+}(t^{-1}Y^{\frac{\alpha^\vee}{2}} - tY^{-\frac{\alpha^\vee}{2}})\Big|_{\Lambda_{n,t}^W}$$
$$= w_0\prod_{\alpha\in R^+}(t^{-1}Y^{\frac{\alpha^\vee}{2}} - tY^{-\frac{\alpha^\vee}{2}})\sum_{w\in W}Y^{w(\omega_n^\vee)}\Big|_{\Lambda_{n,t}^W} \tag{24}$$

turns to be

$$\left\{\sum_{\omega^\vee\in\Omega^\vee}C(\omega^\vee)\tau(\omega^\vee)\right\}w_0\prod_{\alpha\in R^+}(t^{-1}Y^{\frac{\alpha^\vee}{2}} - tY^{-\frac{\alpha^\vee}{2}})\Big|_{\Lambda_{n,t}^W}$$
$$= w_0\prod_{\alpha\in R^+}(t^{-1}Y^{\frac{\alpha^\vee}{2}} - tY^{-\frac{\alpha^\vee}{2}})\sum_{w\in W}Y^{w(\omega_n^\vee)}\Big|_{\Lambda_{n,t}^W}. \tag{25}$$

Now we will determine the coefficients $C(\omega^\vee)$ as follows. Note that we can write

$$w_0\sum_{w\in W}Y^{w(\omega_n^\vee)} \tag{26}$$
$$= \prod_{\alpha\in\hat{R}}\frac{t_\alpha X^\alpha - t_\alpha^{-1}}{X^\alpha - 1}\tau(\omega_n^\vee) + \sum_{\omega^\vee\in\Omega^\vee\setminus\{\omega_n^\vee\}}A(x_1,\ldots,x_n)\tau(\omega^\vee)S,$$

where $A(x_1,\ldots,x_n)$ and S are appropriate coefficients and elements of Weyl group respectively. Substituting Equation (26) to (24), multiplying $w\in W$ to the both sides of (24), and comparing with (25), we can observe that the term of $C(w(\omega_n^\vee))$ is

$$\prod_{\alpha\in\hat{R}} \frac{t_\alpha X^{w(\alpha)} - t_\alpha^{-1}}{X^{w(\alpha)} - 1} \prod_{\beta\in\hat{R}_w^-} \frac{t_\beta X^\beta - t_\beta^{-1}}{t_\beta^{-1} X^\beta - t_\beta} \frac{q^{-2} t_\beta^{-1} X^\beta - t_\beta}{q^{-2} t_\beta X^\beta - t_\beta^{-1}} \tau(w(\omega_n^\vee)). \tag{27}$$

Thus the left hand side of Equation (20) is proved. Hence the proof has completed. □

From Equation (20), one can derive the spin-shift relation. Put $t_\alpha = q^{l_\alpha}$, and conjugating by the weight function Δ^+, namely multiplying the both sides of the above equation by Δ^+ from the left and $(\Delta^+)^{-1}$ from the right, we obtain

$$\begin{aligned}
&\sum_{w\in W} \prod_{\alpha\in R^+} \frac{q^{2(l_\alpha+1)} X^{w(\alpha)} - 1}{q^2 X^{w(\alpha)} - 1} \prod_{\beta\in\hat{R}_w^+} \frac{q^2 X^\beta - 1}{q^{2(l_\beta+1)} X^\beta - 1} \prod_{\gamma\in\hat{R}_w^-} \frac{q^{2l_\gamma} X^{w(\gamma)} - 1}{X^{w(\gamma)} - 1} \tau(w(\omega_n^\vee)) \\
&\times \Delta^+ \hat{D}_{(l_\alpha)} (\Delta^+)^{-1} = \Delta^+ \hat{D}_{(l_\alpha)} (\Delta^+)^{-1} \\
&\times \sum_{w\in W} \prod_{\alpha\in R^+} \frac{q^{2l_\alpha} X^{w(\alpha)} - 1}{X^{w(\alpha)} - 1} \prod_{\beta\in\hat{R}_w^+} \frac{X^\beta - 1}{q^{2l_\beta} X^\beta - 1} \prod_{\gamma\in\hat{R}_w^-} \frac{q^{2l_\gamma - 2} X^{w(\gamma)} - 1}{q^{-2} X^{w(\gamma)} - 1} \tau(w(\omega_n^\vee)).
\end{aligned}$$

where

$$\begin{aligned}
\hat{D}_{(l_\alpha)} &= \prod_{\alpha\in R^+} \frac{t_\alpha X^{w(\alpha)} - t_\alpha^{-1}}{X^\alpha - 1} \\
&\times \sum_{w\in W} \mathrm{sgn}(w) w \left(\prod_{i=2}^{\infty} \prod_{\substack{\alpha\in R^+ \\ ht(\alpha)\geq i}} \frac{t_\alpha q^{2(i-1)} X^\alpha - t_\alpha^{-1}}{q^{2(i-1)X^\alpha - 1}} \tau(\rho^\vee) + \sum_{\lambda^\vee \prec \rho^\vee} G_{\lambda^\vee} \tau(\lambda^\vee) \right).
\end{aligned} \tag{28}$$

Now we calculate the explicit form of the spin-shift operator

$$D_{(l_\alpha)} = \Delta^+ \hat{D}_{(l_\alpha)} (\Delta^+)^{-1}. \tag{29}$$

Since the action of $\tau(w(\rho^\vee))$ to $(X^{w(\alpha)}; q^2)_{l_\alpha}$ is

$$\tau(w(\rho^\vee))(X^{w(\alpha)}; q^2)_{l_\alpha} = (q^{2ht(\alpha)} X^{w(\alpha)}; q^2)_{l_\alpha}, \tag{30}$$

we find

$$\begin{aligned}
&\tau(w(\rho^\vee))(\Delta^+)^{-1} \\
&\quad = \prod_{\alpha\in\hat{R}_w^+} \prod_{i=0}^{l_\alpha - 1} \frac{1}{1 - q^{2(i+ht(\alpha))} X^{w(\alpha)}} \prod_{\alpha\in\hat{R}_w^-} \prod_{i=0}^{l_\alpha - 1} \frac{1}{1 - q^{2(i-ht(\alpha))} X^{-w(\alpha)}} \tau(w(\rho^\vee)).
\end{aligned}$$

In conclusion we obtain the following result.

Theorem 1 *The explicit form of the spin-shift operator for the Ruijsenaars-Schneider Hamiltonian of type C_n is*

$$\begin{aligned} D_{(l_\alpha)} &= \sum_{w\in W} \mathrm{sgn}(w)\Bigg(\prod_{\alpha\in\hat{R}_w^-} \frac{1-q^{2l_\alpha}X^\alpha}{1-X^\alpha}\frac{1-q^{2(l_\alpha-1)}X^\alpha}{1-q^{-2}X^\alpha} \\ &\quad \times \prod_{s=1}^{2n-1} \prod_{\substack{\alpha\in\hat{R}_w^- \\ s+1\leq ht(-w^{-1}(\alpha))}} \frac{1-q^{2(l_\alpha+s)}X^{-\alpha}}{1-q^{2s}X^{-\alpha}}\frac{1-q^{2(l_\alpha-s-1)}X^\alpha}{1-q^{-2(s+1)}X^\alpha}\tau(w(\rho^\vee)) \\ &\quad + \sum_{\lambda^\vee\prec\rho^\vee} G_{\lambda^\vee}\tau(\lambda^\vee)\Bigg). \end{aligned}$$

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