A NEW DESCRIPTION OF CONVEX BASES OF PBW TYPE FOR UNTWISTED QUANTUM AFFINE ALGEBRAS

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Abstract. In [8], we classified all "convex orders" on the positive root system Δ_+ of an arbitrary untwisted affine Lie algebra $\mathfrak{g}$ and gave a concrete method of constructing all convex orders on Δ_+. The aim of this paper is to give a new description of "convex bases" of PBW type of the positive subalgebra U_q^+ of the quantum affine algebra $U_q(\mathfrak{g})$ by using the concrete method of constructing all convex orders on Δ_+. Applying convex properties of the convex bases of U_q^+, for each convex order on Δ_+, we construct a pair of the dual bases of U_q^+ and the negative subalgebra U_q^- with respect to a q-analogue of the Killing form.

1. Introduction

In the theory of quantum algebras, it is an important problem to construct the dual bases of the positive subalgebra U_q^+ and the negative subalgebra U_q^- of the quantum algebra U_q with respect to the q-analogue of the Killing form which is defined in [12] and [15]. For example, the dual bases of U_q^+ and U_q^- were applied to express the universal R-matrix and the extremal projector of U_q in an explicit formula ([12],[13]), and it is known that the dual bases are related to the canonical bases of U_q^+ or the global crystal bases of U_q^- ([3]). The positive and negative parts of the dual bases used to be constructed as a kind of Poincaré-Birkhoff-Witt (PBW) type bases of U_q^+ and U_q^- respectively, and the both parts have several convex properties concerning the q-commutator and the coproduct of U_q. We would like to emphasis that the convex properties are useful for calculating values of the q-Killing form, so we call the positive or negative parts of the dual bases *convex bases* of U_q^+ or U_q^- respectively.

By the way, each convex basis of U_q^+ is formed by monomials in certain q-root vectors E_α with α positive roots multiplied in a predetermined total order on the positive root system Δ_+ of the underlying Lie algebra $\mathfrak{g}$. The total order on Δ_+ has several convex properties, so we call such a total order on Δ_+ "convex order" on Δ_+.

In the case where $\mathfrak{g}$ is an arbitrary finite dimensional simple Lie algebra, there is a natural bijective mapping between the set of the convex orders on Δ_+ and the set of the reduced expressions of the longest element of the Weyl group, and G. Lusztig constructed convex bases of U_q^+ associated with all reduced expressions of the longest element of the Weyl group by using a braid group action on $U_q(\mathfrak{g})$ ([14]). Therefore all convex bases of U_q^+ had been constructed in the finite case.

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In the case where $\mathfrak{g}$ is an arbitrary untwisted affine Lie algebra, in [2], J. Beck constructed convex bases of U_q^+ associated with convex orders on Δ_+ of a special type. On the other hand, in [8], we classified all convex orders on Δ_+, and we found out that there exist new types of convex orders on Δ_+ which was not used in the Beck's construction, and then we gave a concrete method of constructing all convex orders on Δ_+ for the untwisted affine case. So we think that it is natural to extend the Beck's construction of convex bases of U_q^+ by using the new knowledge about convex orders on Δ_+.

In this article, we give a new description of convex bases of U_q^+ for the quantum affine algebra $U_q(\mathfrak{g})$. More precisely, we construct convex bases of U_q^+ by using the concrete method of constructing all convex orders on Δ_+ in the case where $\mathfrak{g}$ is an arbitrary untwisted affine Lie algebra.

We would like to explain more details of our results. Let $\mathfrak{g}$ be an arbitrary untwisted affine Lie algebra, i.e., the affine Lie algebra of type $X_r^{(1)}$, where $X = A, B, C, D, E, F, G$. Let Δ be the root system of $\mathfrak{g}$, $\Pi = \{\alpha_i \,|\, i \in \mathbf{I}\}$ the root basis, and $W = \langle s_i \,|\, i \in \mathbf{I}\rangle$ the Weyl group. Define Δ_+ to be the positive root system with respect to Π, Δ_+^{re} the positive real root system, and Δ_+^{im} the positive imaginary root system. We call a total order $\preceq$ on a subset $B \subset \Delta_+$ a *convex order* on B if the order $\preceq$ satisfies the following two conditions:

$$\text{CO(i)}: \quad (\beta, \gamma) \in B^2 \setminus (\Delta_+^{im})^2, \;\; \beta \prec \gamma, \;\; \beta + \gamma \in B \implies \beta \prec \beta + \gamma \prec \gamma;$$
$$\text{CO(ii)}: \quad \beta \in B, \;\; \gamma \in \Delta_+ \setminus B, \;\; \beta + \gamma \in B \implies \beta \prec \beta + \gamma.$$

Here we write $\beta \prec \gamma$ if $\beta \preceq \gamma$ and $\beta \neq \gamma$. In addition, we denote by $\preceq^{op}$ the total order on B defined by setting $\beta \preceq^{op} \gamma \Leftrightarrow \beta \succeq \gamma$ for each pair $(\beta, \gamma) \in B^2$, and call $\preceq^{op}$ the opposite of $\preceq$. We also say that $\preceq$ is an *opposite convex order* if the opposite $\preceq^{op}$ is a convex order.

We put $\mathring{\mathbf{I}} := \{1, \cdots, r\}$. Then we can express the index set $\mathbf{I}$ of Π by $\mathbf{I} = \{0\} \amalg \mathring{\mathbf{I}}$, and we may regard the subset $\mathring{\Pi} := \{\alpha_i \,|\, i \in \mathring{\mathbf{I}}\}$ of Π as a root basis of the root system $\mathring{\Delta}$ of the underlying Lie subalgebra $\mathring{\mathfrak{g}}$ of type X_r. Let $\mathring{\Delta}_+$ (resp. $\mathring{\Delta}_-$) be the positive root system of $\mathring{\Delta}$ with respect to $\mathring{\Pi}$, and $\mathring{W} := \langle\, s_i \,|\, i \in \mathring{\mathbf{I}} \,\rangle \subset W$ the Weyl group of $\mathring{\mathfrak{g}}$. For each $w \in \mathring{W}$, we set

$$\Delta(w, \pm) := \{\, m\delta + \varepsilon \mid m \in \mathbb{Z}_{\geq 0},\; \varepsilon \in w\mathring{\Delta}_\pm \,\} \cap \Delta_+,$$

where δ is the lowest positive imaginary root. Then we have

$$\Delta_+ = \Delta(w, -) \amalg \Delta_+^{im} \amalg \Delta(w, +).$$

For each subset $\mathbf{J} \subset \mathring{\mathbf{I}}$, we set $\mathring{\Pi}_{\mathbf{J}} := \{\alpha_j \mid j \in \mathbf{J}\}$, $\mathring{W}_{\mathbf{J}} := \langle s_j \mid j \in \mathbf{J}\rangle$, $\mathring{S}_{\mathbf{J}} := \{s_j \mid j \in \mathbf{J}\}$, $\mathring{\Delta}_{\mathbf{J}} := \mathring{W}_{\mathbf{J}}(\mathring{\Pi}_{\mathbf{J}})$. The set $\mathring{\Delta}_{\mathbf{J}}$ is a finite root system with $\mathring{\Pi}_{\mathbf{J}}$ a root basis. Then we define naturally an untwisted "affinization" $(\Pi_{\mathbf{J}}, W_{\mathbf{J}}, S_{\mathbf{J}}, \Delta_{\mathbf{J}}^{re})$ of the quadruplet $(\mathring{\Pi}_{\mathbf{J}}, \mathring{W}_{\mathbf{J}}, \mathring{S}_{\mathbf{J}}, \mathring{\Delta}_{\mathbf{J}})$, where $\mathring{\Pi}_{\mathbf{J}} \subset \Pi_{\mathbf{J}} \subset \Delta_{\mathbf{J}}^{re} \subset \Delta^{re}$ and $\mathring{S}_{\mathbf{J}} \subset S_{\mathbf{J}} \subset W_{\mathbf{J}} \subset W$, and introduce a "subroot system" $\Delta_{\mathbf{J}}$ of Δ by setting $\Delta_{\mathbf{J}} := \Delta_{\mathbf{J}}^{re} \amalg \Delta^{im}$. The sets $S_{\mathbf{J}}$ and $\Pi_{\mathbf{J}}$ are indexed by each other in such a way that $S_{\mathbf{J}} = \{s_\alpha \,|\, \alpha \in \Pi_{\mathbf{J}}\}$ and $\Pi_{\mathbf{J}} = \{\alpha_s \,|\, s \in S_{\mathbf{J}}\}$, where s_α is the reflection with respect to α and $\alpha_s = \alpha$ if and only if $s = s_\alpha$ for $\alpha \in \Pi_{\mathbf{J}}$ and $s \in S_{\mathbf{J}}$. In the case where $\mathbf{J} \neq \emptyset$, the pair $(W_{\mathbf{J}}, S_{\mathbf{J}})$ is an infinite Coxeter group. We denote by $\boldsymbol{s} = (\boldsymbol{s}(p))_{p \in \mathbb{N}}$ an infinite sequence consisting of elements $\boldsymbol{s}(p) \in S_{\mathbf{J}}$ for $p \in \mathbb{N}$, and call such a sequence $\boldsymbol{s}$ an *infinite* *reduced word* of $(W_{\mathbf{J}}, S_{\mathbf{J}})$ if the length of the element $\boldsymbol{s}(1)\boldsymbol{s}(2)\cdots\boldsymbol{s}(p) \in W_{\mathbf{J}}$ is p for each $p \in \mathbb{N}$, and define $\mathcal{W}_{\mathbf{J}}^{\infty}$ to be the subset of $S_{\mathbf{J}}^{\mathbb{N}}$ consisting of all infinite reduced words of $(W_{\mathbf{J}}, S_{\mathbf{J}})$. For each $\boldsymbol{s} = (\boldsymbol{s}(p))_{p\in\mathbb{N}} \in \mathcal{W}_{\mathbf{J}}^{\infty}$, we obtain an injective mapping $\phi_{\boldsymbol{s}} : \mathbb{N} \to \Delta_{\mathbf{J}+}^{re}$ by setting

$$\phi_{\boldsymbol{s}}(p) := \boldsymbol{s}(1)\cdots\boldsymbol{s}(p-1)(\alpha_{\boldsymbol{s}(p)}),$$

and denote by $\Phi_{\mathbf{J}}^{\infty}([\boldsymbol{s}])$ the image of the injective mapping $\phi_{\boldsymbol{s}}$. Let $\mathring{W}/\mathring{W}_{\mathbf{J}}$ be the set of the left cosets, and $\mathring{W}^{\mathbf{J}}$ the set of the minimal coset representatives of elements of $\mathring{W}/\mathring{W}_{\mathbf{J}}$. Then each element $w \in \mathring{W}$ can be uniquely written as $w = w^{\mathbf{J}}w_{\mathbf{J}}$ with $w^{\mathbf{J}} \in \mathring{W}^{\mathbf{J}}$ and $w_{\mathbf{J}} \in \mathring{W}_{\mathbf{J}}$.

The following is a summary of the results of the paper [8].

Theorem 1.1. *Let w be an arbitrary element of $\mathring{W}$.*

(1) *Let $\preceq_-$ be an arbitrary convex order on $\Delta(w,-)$, $\preceq_0$ an arbitrary total order on Δ_+^{im}, and $\preceq_+$ an arbitrary opposite convex order on $\Delta(w,+)$. Then we can define a convex order $\preceq$ on Δ_+ by extending $\preceq_-, \preceq_0, \preceq_+$ to on $\Delta_+ = \Delta(w,-) \amalg \Delta_+^{im} \amalg \Delta(w,+)$ in such a way that*

$$\Delta(w,-) \prec \Delta_+^{im} \prec \Delta(w,+).$$

Moreover, we can obtain every convex orders on Δ_+ by applying the procedure above.

(2) *For each positive integer $n \le r = \sharp\mathring{\mathbf{I}}$ and sequence $\mathbf{J}_\bullet = (\mathbf{J}_0, \mathbf{J}_1, \ldots, \mathbf{J}_n)$ consisting of subsets of $\mathring{\mathbf{I}}$ such that $\mathring{\mathbf{I}} = \mathbf{J}_0 \supsetneq \mathbf{J}_1 \supsetneq \cdots \supsetneq \mathbf{J}_n = \emptyset$, there exist $y_\bullet = (y_1, \ldots, y_n) \in W_{\mathbf{J}_1} \times \cdots \times W_{\mathbf{J}_n}$ and $\boldsymbol{s}_\bullet = (\boldsymbol{s}_0, \ldots, \boldsymbol{s}_{n-1}) \in \mathcal{W}_{\mathbf{J}_0}^{\infty} \times \cdots \times \mathcal{W}_{\mathbf{J}_{n-1}}^{\infty}$ which satisfy*

$$\Delta(w,-) = \amalg_{i=1}^{n} w^{\mathbf{J}_{i-1}} y_{i-1} \Phi_{\mathbf{J}_{i-1}}^{\infty}([\boldsymbol{s}_{i-1}]), \tag{1.1}$$

where $y_0 := 1$, and then we can define a convex order $\preceq$ on $\Delta(w,-)$ by applying the following procedure Steps 1,2.

Step 1. *For each $i = 1, \ldots, n$, define a total order $\preceq_i$ on the following set*

$$R_i := w^{\mathbf{J}_{i-1}} y_{i-1} \Phi_{\mathbf{J}_{i-1}}^{\infty}([\boldsymbol{s}_{i-1}])$$

by setting

$$w^{\mathbf{J}_{i-1}} y_{i-1} \phi_{\boldsymbol{s}_{i-1}}(p) \preceq_i w^{\mathbf{J}_{i-1}} y_{i-1} \phi_{\boldsymbol{s}_{i-1}}(q) \quad \textit{for each } p \le q.$$

Step 2. *Define $\preceq$ by extending $\preceq_1, \ldots, \preceq_n$ to on $\Delta(w,-) = \amalg_{i=1}^{n} R_i$ in such a way that*

$$R_i \prec R_{i'} \quad \textit{for each } i < i'.$$

Conversely, for each convex order on $\preceq$ on $\Delta(w,-)$, there exists a unique quadruplet $(n, \mathbf{J}_\bullet, y_\bullet, \boldsymbol{s}_\bullet)$ which satisfies the condition (1.1) and the convex order $\preceq$ can be constructed by the procedure Steps 1,2 above.

We remark that Theorem 1.1 gives a concrete method of constructing all convex order on Δ_+, since $\Delta(w,+) = \Delta(ww_\circ,-)$ with $w_\circ$ the longest element of $\mathring{W}$. For each positive integer $n \le r = \sharp\mathring{\mathbf{I}}$, we call the convex order on $\Delta(w,-)$ described above that of *n-row type*.

We denote simply by U the quantized enveloping algebra $U_q(\mathfrak{g})$ over $\mathbb{Q}(q)$ for an arbitrary affine Lie algebra $\mathfrak{g}$ of type $X_r^{(1)}$, where $X = A, B, C, D, E, F, G$. For each non-empty subset $\mathbf{J} \subset \mathring{\mathbf{I}}$, we introduce the $\mathbb{Q}(q)$-subalgebra

$$U_{\mathbf{J}} = \langle E_\alpha, K_\alpha^{\pm 1}, F_\alpha \mid \alpha \in \Pi_{\mathbf{J}} \rangle$$

of U with $\Delta_{\mathbf{J}}$ the root system, which can be regarded as an untwisted "affinization" of the $\mathbb{Q}(q)$-subalgebra generated by $\{E_\alpha, K_\alpha^{\pm 1}, F_\alpha \mid \alpha \in \mathring{\Pi}_{\mathbf{J}}\}$. Let $\mathcal{B}_{W_{\mathbf{J}}} = \langle {}_{\mathbf{J}}T_y \mid y \in W_{\mathbf{J}} \rangle$ be the braid group associated with the Coxeter group $(W_{\mathbf{J}}, S_{\mathbf{J}})$. We define an action of $\mathcal{B}_{W_{\mathbf{J}}}$ on the subalgebra $U_{\mathbf{J}}$, i.e., we construct a group homomorphism $\mathcal{B}_{W_{\mathbf{J}}} \to \mathrm{Aut}(U_{\mathbf{J}})$, which is a natural generalization of the well-known braid group action $\mathcal{B}_W \to \mathrm{Aut}(U)$ introduced by G. Lusztig. Let U^+ be the positive subalgebra of U, i.e, the $\mathbb{Q}(q)$-subalgebra generated by $\{E_\alpha \mid \alpha \in \Pi\}$. Let $\mathcal{A}_1$ be the $\mathbb{Q}$-subalgebra of $\mathbb{Q}(q)$ consisting of elements of $\mathbb{Q}(q)$ which have no pole at $q = 1$, and ${}_{\mathcal{A}_1}U^+$ the $\mathcal{A}_1$-subalgebra of U generated by $\{E_\alpha \mid \alpha \in \Pi\}$.

The following is the main result of this paper, which is a part of the results of Theorem 8.6 in the case where $\mathbf{J} = \mathring{\mathbf{I}}$.

Theorem 1.2. *Let $\preceq$ be an arbitrary convex order on Δ_+, and $w \in \mathring{W}$ a unique element such that*

$$\Delta(w,-) \prec \Delta_+^{im} \prec \Delta(w,+).$$

We define $\preceq_-$, $\preceq_0$, and $\preceq_+$ to be the restriction of $\preceq$ to on $\Delta(w,-)$, on Δ_+^{im}, and on $\Delta(w,+)$, respectively, and define a total order $\tilde{\preceq}_0$ on the following set

$$\tilde{\Delta}_+^{im} := \Delta_+^{im} \times \mathring{\mathbf{I}} = \{ (k\delta, i) \mid k \in \mathbb{N},\ i \in \mathring{\mathbf{I}} \}$$

by setting

$$(k\delta, i)\ \tilde{\prec}_0\ (k'\delta, i') \Longleftrightarrow \begin{cases} k\delta\ \prec_0\ k'\delta & \text{if } k \neq k', \\ i < i' & \text{if } k = k'. \end{cases}$$

In addition, we define a total order $\tilde{\preceq}$ on the following set

$$\tilde{\Delta}_+ := \Delta_+^{re} \amalg \tilde{\Delta}_+^{im} = \Delta(w,-) \amalg \tilde{\Delta}_+^{im} \amalg \Delta(w,+)$$

by extending $\preceq_-$, $\tilde{\preceq}_0$, and $\preceq_+$ such as

$$\Delta(w,-)\ \tilde{\prec}\ \tilde{\Delta}_+^{im}\ \tilde{\prec}\ \Delta(w,+).$$

Let $(n, \mathbf{J}_\bullet, y_\bullet, \boldsymbol{s}_\bullet)$ be a unique quadruplet satisfying the condition (1.1) in Theorem 1.1 which corresponds to the convex order $\preceq_-$ on $\Delta(w,-)$. For each $\eta \in \Delta(w,-)$, we define a weight vector $E_{\preceq_-,\eta} \in U^+$ with weight η by setting

$$E_{\preceq_-,\eta} := T_{w^{\mathbf{J}_{i-1}}} \cdot {}_{\mathbf{J}_{i-1}}T_{y_{i-1}}(E_{\boldsymbol{s}_{i-1}(p)}) \tag{1.2}$$

in the case where $\eta = w^{\mathbf{J}_{i-1}} y_{i-1} \phi_{\boldsymbol{s}_{i-1}}(p)$ with $i \in \mathbb{N}_n$ and $p \in \mathbb{N}$. Here, $E_{\boldsymbol{s}_{i-1}(p)} = E_\alpha$ if $\alpha_{\boldsymbol{s}_{i-1}(p)} = \alpha \in \Pi_{\mathbf{J}_{i-1}}$. In addition, for each $\eta \in \tilde{\Delta}_+$, we set

$$E_{\preceq,\eta} := \begin{cases} E_{\preceq_-,\eta} & \text{if } \eta \in \Delta(w,-), \\ T_w(I_{i,k}) & \text{if } \eta = (k\delta, i) \in \tilde{\Delta}_+^{im}, \\ \Psi(E_{\preceq_+^{op},\eta}) & \text{if } \eta \in \Delta(w,+), \end{cases}$$

where $\preceq_+^{op}$ is the opposite order of $\preceq_+$, $E_{\preceq_+^{op},\eta}$ is defined by applying (1.2) for the convex order $\preceq_+^{op}$ on $\Delta(w,+)$, Ψ is a $\mathbb{Q}(q)$-algebra anti-automorphism of U, and $I_{i,k}$

is a weight vector with weight $k\delta$ for each $(i,k) \in \mathring{\mathbf{I}} \times \mathbb{N}$. Let $E_{\prec}(\tilde{\Delta}_+)$ be the set of all monomials

$$E_{\preceq,\eta_1}^{c_1} E_{\preceq,\eta_2}^{c_2} \cdots E_{\preceq,\eta_m}^{c_m}$$

of the set $\{E_{\preceq,\eta} \,|\, \eta \in \tilde{\Delta}_+\}$ multiplied in the order $\tilde{\preceq}$, where $\eta_1 \,\tilde{\prec}\, \eta_2 \,\tilde{\prec} \cdots \tilde{\prec}\, \eta_m$ and $\mathbf{c} = (c_1, c_2, \ldots, c_m) \in (\mathbb{Z}_{\geq 0})^m$ with $m \in \mathbb{N}$ arbitrary. Then the set $E_{\prec}(\tilde{\Delta}_+)$ is a basis of the $\mathbb{Q}(q)$-subalgebra U^+ and of the $\mathcal{A}_1$-subalgebra ${}_{\mathcal{A}_1}U^+$. Moreover, the following equality holds:

$$[E_{\preceq,\eta}\,,\,E_{\preceq,\zeta}]_q = \sum_{\eta \tilde{\prec} \eta_1 \tilde{\prec} \eta_2 \tilde{\prec} \cdots \tilde{\prec} \eta_m \tilde{\prec} \zeta} h_{\mathbf{c}} E_{\preceq,\eta_1}^{c_1} E_{\preceq,\eta_2}^{c_2} \cdots E_{\preceq,\eta_m}^{c_m}$$

for each $\eta, \zeta \in \tilde{\Delta}_+$ satisfying $\eta \,\tilde{\prec}\, \zeta$, where $h_{\mathbf{c}} \in \mathcal{A}_1$ and $[\ ,\ ]_q$ is the q-commutator.

We note that in [2] J. Beck constructed convex bases of U_q^+ associated with convex orders $\preceq_-$ on $\Delta(1,-)$ of 1-row type and opposite convex orders $\preceq_+$ on $\Delta(1,+)$ of 1-row type.

This paper is organized as follows. In section 2, we give notations and preliminary results for the root system of the untwisted affine Lie algebra $\mathfrak{g}$. In section 3, we give notations and preliminary results for reduced words of the Coxeter group $(W_{\mathbf{J}}, S_{\mathbf{J}})$ and convex orders on the positive root system $\Delta_{\mathbf{J}+}$. In section 4, we give notations and preliminary results for the quantum algebra. In section 5, we construct the subalgebra $U_{\mathbf{J}}$ of $U_q(\mathfrak{g})$ associated with $\Delta_{\mathbf{J}}$ and the braid group action on it. In section 6, we define imaginary root vectors of $U_{\mathbf{J}}^+$. In section 7, we give several tensor product decompositions of the positive subalgebra $U_{\mathbf{J}}^+$ of $U_{\mathbf{J}}$. In section 8, we give a concrete method of constructing convex bases of $U_{\mathbf{J}}^+$ associated with arbitrary convex orders on $\Delta_{\mathbf{J}+}$. In section 9, we construct the dual convex bases of U^+ and U^- with respect to the q-Killing form, and then present the multiplicative formula for the R-matrix of $U_q(\mathfrak{g})$ associated with an arbitrary convex order on Δ_+.

2. Notations and preliminary results for the untwisted affine root systems

Let $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ be the set of the rational numbers, the integers, and the positive integers, respectively, and let $\mathbb{Z}_+$ and $\mathbb{Z}_-$ be the set of the non-negative integers and the non-positive integers, respectively, i.e., $\mathbb{Z}_+ = \mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_- = \mathbb{Z}_{\leq 0}$. We denote by $\sharp S$ the cardinality of a set S, and write $\sharp S = \infty$ if S is an infinite set.

Let $\mathrm{A} = [\mathrm{A}_{ij}]_{i,j \in \mathbf{I}}$ be an arbitrary symmetrizable generalized Cartan matrix with $\mathbf{I}$ an index set, and $(d_i)_{i \in \mathbf{I}}$ be relatively prime positive integers such that the matrix $[d_i \mathrm{A}_{ij}]_{i,j \in \mathbf{I}}$ is symmetric. Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a minimal realization of A over $\mathbb{Q}$, that is, a triplet consisting of a $(\sharp\mathbf{I} + \mathrm{corank}\mathrm{A})$-dimensional vector space $\mathfrak{h}$ over $\mathbb{Q}$ and linearly independent subsets $\Pi = \{\alpha_i \,|\, i \in \mathbf{I}\} \subset \mathfrak{h}^*$, $\Pi^\vee = \{\alpha_i^\vee \,|\, i \in \mathbf{I}\} \subset \mathfrak{h}$ satisfying $\langle \alpha_i^\vee, \alpha_j \rangle = \mathrm{A}_{ij}$ for $i, j \in \mathbf{I}$. Here, $\mathfrak{h}^*$ is the dual space of $\mathfrak{h}$ and the map $\langle \cdot, \cdot \rangle \colon \mathfrak{h} \times \mathfrak{h}^* \to \mathbb{Q}$ is the canonical paring. Let $\mathfrak{g} = \langle e_i, h, f_i \,|\, i \in \mathbf{I}, h \in \mathfrak{h} \rangle$ be the Kac-Moody Lie algebra associated with $(\mathfrak{h}, \Pi, \Pi^\vee)$, $\Delta \subset \mathfrak{h}^* \setminus \{0\}$ the root system, Δ^{re} (resp. Δ^{im}) the real (resp. imaginary) root system, and $W = \langle\, s_i \,|\, i \in \mathbf{I} \,\rangle \subset \mathrm{GL}(\mathfrak{h}^*)$ the Weyl group. Here, s_i is the reflection with respect to α_i acting on $\mathfrak{h}^*$ by $s_i(\lambda) = \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i$ for $\lambda \in \mathfrak{h}^*$. Set $S := \{s_i \,|\, i \in \mathbf{I}\}$ then the pair (W, S) is a Coxeter system. Let $\ell \colon W \to \mathbb{Z}_+$ be the length function of the Coxeter system. Let

Δ_+ (resp. Δ_-) be the positive (resp. negative) root system with respect to the root basis Π, and set $\Delta^{re}_\pm = \Delta^{re} \cap \Delta_\pm$ and $\Delta^{im}_\pm = \Delta^{im} \cap \Delta_\pm$. We set

$$\mathfrak{h}'^* := \oplus_{i\in\mathbf{I}}\mathbb{Q}\alpha_i, \quad Q := \oplus_{i\in\mathbf{I}}\mathbb{Z}\alpha_i, \quad Q_\pm := \textstyle\sum_{i\in\mathbf{I}}\mathbb{Z}_\pm\alpha_i,$$
$$P := \{\, \lambda \in \mathfrak{h}^* \mid \langle\alpha_i^\vee, \lambda\rangle \in \mathbb{Z} \quad (\forall i \in \mathbf{I})\,\}.$$

Fix a subspace $\mathfrak{h}''^* \subset \mathfrak{h}^*$ such that $\mathfrak{h}^* = \mathfrak{h}'^* \oplus \mathfrak{h}''^*$, and define a non-degenerate symmetric bilinear form $(\cdot\,|\,\cdot)$ on $\mathfrak{h}^*$ by setting

$$(\alpha_i\,|\,\lambda) := d_i\langle\alpha_i^\vee, \lambda\rangle \quad (i \in \mathbf{I}, \lambda \in \mathfrak{h}^*), \quad (\lambda\,|\,\mu) := 0 \quad (\lambda, \mu \in \mathfrak{h}''^*).$$

Note that $(\alpha_i\,|\,\alpha_j) = d_i\mathrm{A}_{ij}$ and $(\alpha_i\,|\,\lambda) \in \mathbb{Z}$ for $i, j \in \mathbf{I}$ and $\lambda \in P$. We denote by $\mathrm{O}(\mathfrak{h}^*)$ the orthogonal group with respect to the form $(\cdot\,|\,\cdot)$. The reflection $s_\alpha \in \mathrm{O}(\mathfrak{h}^*)$ with respect to $\alpha \in \Delta^{re}$ acts on $\mathfrak{h}^*$ by $s_\alpha(\lambda) = \lambda - \frac{2(\alpha\,|\,\lambda)}{(\alpha\,|\,\alpha)}\alpha$ for $\lambda \in \mathfrak{h}^*$. We set

$$\mathrm{Aut}(\Delta) := \{\phi \in \mathrm{O}(\mathfrak{h}^*)\,|\,\phi(\Delta) = \Delta\}, \quad \mathrm{Aut}(\Delta, \Pi) := \{\phi \in \mathrm{Aut}(\Delta)\,|\,\phi(\Pi) = \Pi\},$$

and identify the group $\mathrm{Aut}(\Delta, \Pi)$ with a permutation group of $\mathbf{I}$ by $\rho(i) = j$ if $\rho(\alpha_i) = \alpha_j$ for $\rho \in \mathrm{Aut}(\Delta, \Pi)$ and $i, j \in \mathbf{I}$. Then $\rho s_i = s_{\rho(i)}\rho$ and $\mathrm{ord}(s_is_j) = \mathrm{ord}(s_{\rho(i)}s_{\rho(j)})$, where $\mathrm{ord}(x)$ is the order of x.

Throughout this section from now on, we assume that $\mathrm{A} = [\mathrm{A}_{ij}]_{i,j\in\mathbf{I}}$ is of the untwisted affine type $X_r^{(1)}$, where $r \in \mathbb{N}$, $X = A, B, \ldots, G$. In addition, we may assume that $\mathbf{I} = \{0, 1, \ldots, r\}$ and $[\mathrm{A}_{ij}]_{i,j\in\mathring{\mathbf{I}}}$ is the Cartan matrix of the finite type X_r with $\mathring{\mathbf{I}} = \{1, \ldots, r\}$. We set

$$\mathring{\Pi} := \{\alpha_i\,|\,i \in \mathring{\mathbf{I}}\}, \quad \mathring{\mathfrak{h}}^* := \mathrm{span}_{\mathbb{Q}}\mathring{\Pi}, \quad \mathring{Q} := \mathrm{span}_{\mathbb{Z}}\mathring{\Pi},$$
$$\mathring{W} := \langle\, s_i\,|\,i \in \mathring{\mathbf{I}}\,\rangle, \quad \mathring{\Delta} := \mathring{W}(\mathring{\Pi}), \quad \mathring{\Delta}_\pm := \mathring{\Delta} \cap \Delta_\pm.$$

Note that $\mathring{\Delta}$ is a root system of type X_r with $\mathring{\Pi}$ a root basis and $\mathring{W}$ the Weyl group. Introduce an element λ_0 of $\mathfrak{h}^*$ such that $\mathfrak{h}^* = \mathfrak{h}'^* \oplus \mathbb{Q}\lambda_0$, and define a non-degenerate symmetric bilinear form $(\cdot\,|\,\cdot)$ on $\mathfrak{h}^*$ by setting $(\alpha_i\,|\,\alpha_j) = d_i\mathrm{A}_{ij}$ for $i, j \in \mathbf{I}$, $(\lambda_0\,|\,\lambda_0) = 0$, and $(\alpha_i\,|\,\lambda_0) = d_0\delta_{i0}$ for $i \in \mathbf{I}$. Denote by θ the highest positive root of $\mathring{\Delta}$ and set $\delta := \alpha_0 + \theta$. Then $(\delta\,|\,\delta) = 0$ and

$$\Delta^{re} = \{\, m\delta + \varepsilon \mid m \in \mathbb{Z},\, \varepsilon \in \mathring{\Delta}\}, \qquad \Delta^{im} = \{\, m\delta \mid m \in \mathbb{Z} \setminus \{0\}\}.$$

Moreover, we see that the form $(\cdot\,|\,\cdot)$ is positive-definite on $\mathring{\mathfrak{h}}^*$ and that the following direct sum is an orthogonal decomposition:

$$\mathfrak{h}^* = \mathring{\mathfrak{h}}^* \oplus (\mathbb{Q}\delta + \mathbb{Q}\lambda_0).$$

For each $\lambda \in \mathfrak{h}^*$, we denote by $\overline{\lambda}$ the image of λ by the orthogonal projection onto $\mathring{\mathfrak{h}}^*$. Then each $\beta \in \Delta$ can be uniquely written as $\beta = m\delta + \overline{\beta}$ with $m \in \mathbb{Z}$ and $\overline{\beta} \in \mathring{\Delta} \amalg \{0\}$. For each $\lambda \in \mathfrak{h}^*$, we define $t_\lambda \in \mathrm{GL}(\mathfrak{h}^*)$ by setting

$$t_\lambda(\mu) = \mu + (\mu\,|\,\delta)\lambda - \{(\mu\,|\,\lambda) + \tfrac{1}{2}(\lambda\,|\,\lambda)(\mu\,|\,\delta)\}\delta \quad (\mu \in \mathfrak{h}^*),$$

which is called the translation with respect to λ. Note that $wt_\lambda = t_{w(\lambda)}w$ for $w \in \mathring{W}$ and that $t_\lambda(\mu) = \mu - (\mu\,|\,\lambda)\delta$ for $\mu \in \mathfrak{h}'^*$. For each $i \in \mathring{\mathbf{I}}$, we set $\check{\alpha}_i := \frac{2\alpha_i}{(\alpha_i\,|\,\alpha_i)}$ and introduce an unique element $\varepsilon_i \in \mathring{\mathfrak{h}}^*$ such that $(\varepsilon_i \,|\, \alpha_j) = \delta_{ij}$ for all $j \in \mathring{\mathbf{I}}$. Note that $\check{\alpha}_i = \sum_{j\in\mathring{\mathbf{I}}} \mathrm{A}_{ij}\varepsilon_j$. We set

$$\mathring{Q}^\vee := \oplus_{i\in\mathring{\mathbf{I}}}\mathbb{Z}\check{\alpha}_i, \quad \mathring{P}^\vee := \oplus_{i\in\mathring{\mathbf{I}}}\mathbb{Z}\varepsilon_i, \quad \mathrm{T} := \{\, t_\lambda \mid \lambda \in \mathring{Q}^\vee \}, \quad \widehat{\mathrm{T}} := \{\, t_\lambda \mid \lambda \in \mathring{P}^\vee \}.$$

Then T is a normal subgroup of W such that $W = \mathrm{T} \rtimes \mathring{W}$. Let us denote the extended Weyl group $\widehat{\mathrm{T}} \rtimes \mathring{W}$ by $\widehat{W}$. For each $x \in \widehat{W}$, there exists an unique element $\overline{x} \in \mathring{W}$ such that $x \in \widehat{\mathrm{T}}\overline{x}$. The assignment $x \mapsto \overline{x}$ defines a group homomorphism $\overline{\cdot}\colon \widehat{W} \to \mathring{W}$. Note that $\overline{x(\lambda)} = \overline{x}(\overline{\lambda})$ and $\overline{s_\alpha} = s_{\overline{\alpha}}$ for $x \in \widehat{W}$, $\lambda \in \mathfrak{h}'^*$, and $\alpha \in \Delta^{re}$. We set $\Omega := \widehat{W} \cap \mathrm{Aut}(\Delta, \Pi)$. Then $\widehat{W} = W \rtimes \Omega$. We set $\widehat{S} := S \amalg (\Omega \setminus \{1\})$ and extend the length function $\ell\colon W \to \mathbb{Z}_+$ of the Coxeter system (W, S) to a $\mathbb{Z}_+$-valued function ℓ on $\widehat{W}$ by setting $\ell(x\rho) := \ell(x)$ for $x \in W$ and $\rho \in \Omega$.

For each subset $\mathbf{J} \subset \mathring{\mathbf{I}}$, we set

$$\mathring{\Pi}_{\mathbf{J}} := \{\alpha_j \,|\, j \in \mathbf{J}\}, \quad \mathring{\mathfrak{h}}^*_{\mathbf{J}} := \mathrm{span}_{\mathbb{Q}}\mathring{\Pi}_{\mathbf{J}} \subset \mathring{\mathfrak{h}}^*, \quad \mathring{Q}_{\mathbf{J}} := \mathrm{span}_{\mathbb{Z}}\mathring{\Pi}_{\mathbf{J}},$$
$$\mathring{W}_{\mathbf{J}} := \langle\, s_j \,|\, j \in \mathbf{J} \,\rangle, \quad \mathring{\Delta}_{\mathbf{J}} := \mathring{W}_{\mathbf{J}}(\mathring{\Pi}_{\mathbf{J}}), \quad \mathring{\Delta}_{\mathbf{J}\pm} := \mathring{\Delta}_{\mathbf{J}} \cap \mathring{\Delta}_{\pm}.$$

Note that $\mathring{\Delta}_{\mathbf{J}}$ is a root system with $\mathring{\Pi}_{\mathbf{J}}$ a root basis and $\mathring{W}_{\mathbf{J}}$ the Weyl group if $\mathbf{J} \neq \emptyset$. We say that $\mathbf{J}$ is *connected* if $\mathring{\Delta}_{\mathbf{J}}$ is an irreducible root system and that a subset $\mathbf{J}' \subset \mathring{\mathbf{I}}$ is *disjointed* with $\mathbf{J}$ if $\mathring{\Delta}_{\mathbf{J}'} \cap \mathring{\Delta}_{\mathbf{J}} = \emptyset$.

For each non-empty subset $\mathbf{J} \subset \mathring{\mathbf{I}}$, let $\mathbf{J}_1, \dots, \mathbf{J}_{\mathrm{C}(\mathbf{J})}$ be the connected components of $\mathbf{J}$ with $\mathrm{C}(\mathbf{J})$ the number of the connected components. Then

$$\mathring{\Delta}_{\mathbf{J}} = \mathring{\Delta}_{\mathbf{J}_1} \amalg \cdots \amalg \mathring{\Delta}_{\mathbf{J}_{\mathrm{C}(\mathbf{J})}}$$

is the irreducible decomposition of $\mathring{\Delta}_{\mathbf{J}}$. For each $c = 1, \dots, \mathrm{C}(\mathbf{J})$, we denote by $\theta_{\mathbf{J}_c}$ the highest positive root of $\mathring{\Delta}_{\mathbf{J}_c}$ relative to the basis $\mathring{\Pi}_{\mathbf{J}_c}$, and set

$$\Pi_{\mathbf{J}_c} := \mathring{\Pi}_{\mathbf{J}_c} \amalg \{\delta - \theta_{\mathbf{J}_c}\}, \quad \Pi_{\mathbf{J}} := \Pi_{\mathbf{J}_1} \amalg \cdots \amalg \Pi_{\mathbf{J}_{\mathrm{C}(\mathbf{J})}}, \quad S_{\mathbf{J}} := \{s_\alpha \,|\, \alpha \in \Pi_{\mathbf{J}}\}.$$

For each element $s \in S_{\mathbf{J}}$, we denote by α_s the unique element of $\Pi_{\mathbf{J}}$ such that $s = s_{\alpha_s}$. Let $W_{\mathbf{J}}$ be the subgroup of W generated by $S_{\mathbf{J}}$, $\mathfrak{h}'^*_{\mathbf{J}}$ the linear subspace of $\mathfrak{h}'^*$ spanned by $\Pi_{\mathbf{J}}$, and $Q_{\mathbf{J}}$ the sublattice of Q spanned by $\Pi_{\mathbf{J}}$ over $\mathbb{Z}$. Note that $\mathfrak{h}'^*_{\mathbf{J}} = \mathring{\mathfrak{h}}^*_{\mathbf{J}} \oplus \mathbb{Q}\delta$, $Q_{\mathbf{J}} = \mathring{Q}_{\mathbf{J}} \oplus \mathbb{Z}\delta$, and $\dim \mathfrak{h}'^*_{\mathbf{J}} = \mathrm{rank}\, Q_{\mathbf{J}} = \sharp\mathbf{J} + 1$. We set

$$\Delta^{re}_{\mathbf{J}} := \{\, \alpha \in \Delta^{re} \mid \overline{\alpha} \in \mathring{\Delta}_{\mathbf{J}} \}, \quad \Delta_{\mathbf{J}} := \Delta^{re}_{\mathbf{J}} \amalg \Delta^{im},$$
$$\Delta^{re}_{\mathbf{J}\pm} := \Delta^{re}_{\mathbf{J}} \cap \Delta_{\pm}, \quad \Delta_{\mathbf{J}\pm} := \Delta_{\mathbf{J}} \cap \Delta_{\pm}, \quad Q_{\mathbf{J}\pm} := Q_{\mathbf{J}} \cap Q_{\pm}.$$

Proposition 2.1 ([7]). *For each non-empty subset $\mathbf{J} \subset \mathring{\mathbf{I}}$, the pair $(W_{\mathbf{J}}, S_{\mathbf{J}})$ is a Coxeter group with $(\mathfrak{h}'^*_{\mathbf{J}}, \Delta_{\mathbf{J}}, \Pi_{\mathbf{J}})$ a root system, namely the triplet has the following properties* (i)–(iv):

(i) *the $\mathfrak{h}'^*_{\mathbf{J}}$ is a representation space of $W_{\mathbf{J}}$ over $\mathbb{Q}$ and the $\Delta_{\mathbf{J}}$ is a subset of $\mathfrak{h}'^*_{\mathbf{J}}$ such that $\Delta_{\mathbf{J}} = -\Delta_{\mathbf{J}}$ and $\Delta_{\mathbf{J}} \setminus \{0\}$ is $W_{\mathbf{J}}$-invariant.*

(ii) *the $\Pi_{\mathbf{J}} = \{\, \alpha_s \,|\, s \in S_{\mathbf{J}} \,\}$ is a subset of $\Delta_{\mathbf{J}}$ such that each element $\alpha \in \Delta_{\mathbf{J}}$ can be written as $\sum_{s\in S} a_s\alpha_s$ with either $a_s \in \mathbb{Z}_+$ for all $s \in S$ or $a_s \in \mathbb{Z}_-$ for all $s \in S$, but not in both ways. Accordingly, we write $\alpha > 0$ or $\alpha < 0$ and set $\Delta_{\mathbf{J}+} := \{\, \alpha \in \Delta_{\mathbf{J}} \,|\, \alpha > 0 \,\}$ and $\Delta_{\mathbf{J}-} := \{\, \alpha \in \Delta_{\mathbf{J}} \,|\, \alpha < 0 \,\}$.*

(iii) *for each* $s \in S_{\mathbf{J}}$*, the equalities* $s(\alpha_s) = -\alpha_s$ *and* $s(\Delta_{\mathbf{J}+} \setminus \{\alpha_s\}) = \Delta_{\mathbf{J}+} \setminus \{\alpha_s\}$ *hold.*

(iv) *if* $w \in W_{\mathbf{J}}$ *and* $s, s' \in S_{\mathbf{J}}$ *satisfy* $w(\alpha_{s'}) = \alpha_s$*, then* $ws'w^{-1} = s$*.*

Remark 2.2. *The action of* $W_{\mathbf{J}}$ *on* $\mathfrak{h}'^*_{\mathbf{J}}$ *is faithful, and hence we may identify* $W_{\mathbf{J}}$ *with a subgroup of* $\mathrm{O}(\mathfrak{h}'^*_{\mathbf{J}})$*. Here,*

$$\mathrm{O}(\mathfrak{h}'^*_{\mathbf{J}}) = \{\, \phi \in \mathrm{GL}(\mathfrak{h}'^*_{\mathbf{J}}) \mid (\phi(\lambda) \,|\, \phi(\mu)) = (\lambda \,|\, \mu) \quad (\lambda, \mu \in \mathfrak{h}'^*_{\mathbf{J}}) \,\}.$$

Let $\ell_{\mathbf{J}} \colon W_{\mathbf{J}} \to \mathbb{Z}_+$ be the length function of the Coxeter group $(W_{\mathbf{J}}, S_{\mathbf{J}})$. We set

$$\mathring{Q}^{\vee}_{\mathbf{J}} := \oplus_{j \in \mathbf{J}} \mathbb{Z}\check{\alpha}_j, \qquad \mathrm{T}_{\mathbf{J}} := \{\, t_\lambda \mid \lambda \in \mathring{Q}^{\vee}_{\mathbf{J}} \,\}.$$

Then $\mathrm{T}_{\mathbf{J}}$ is a normal subgroup of $W_{\mathbf{J}}$ such that $W_{\mathbf{J}} = \mathrm{T}_{\mathbf{J}} \rtimes \mathring{W}_{\mathbf{J}}$. We set

$$\mathring{P}^{\vee}_{\mathbf{J}} := \oplus_{j \in \mathbf{J}} \mathbb{Z}\varepsilon_j, \quad \widehat{\mathrm{T}}_{\mathbf{J}} := \{\, t_\lambda \mid \lambda \in \mathring{P}^{\vee}_{\mathbf{J}} \,\}, \quad \widehat{W}_{\mathbf{J}} := \widehat{\mathrm{T}}_{\mathbf{J}} \rtimes \mathring{W}_{\mathbf{J}} \subset \mathrm{O}(\mathfrak{h}'^*_{\mathbf{J}}).$$

For each $\mathbf{K} \subset \mathbf{J}$, let $\mathring{W}_{\mathbf{J}}/\mathring{W}_{\mathbf{K}}$ be the set of the left cosets, and $\mathring{W}^{\mathbf{K}}_{\mathbf{J}}$ the set of the minimal coset representatives of elements of $\mathring{W}_{\mathbf{J}}/\mathring{W}_{\mathbf{K}}$. If $\mathbf{J} = \mathring{\mathbf{I}}$ we denote it simply by $\mathring{W}^{\mathbf{K}}$. Note that

$$\mathring{W}^{\mathbf{K}}_{\mathbf{J}} = \{\, w \in \mathring{W}_{\mathbf{J}} \mid w(\alpha_k) > 0 \text{ for all } k \in \mathbf{K} \,\}$$

and that each element $w \in \mathring{W}_{\mathbf{J}}$ can be uniquely written as $w = w^{\mathbf{K}} w_{\mathbf{K}}$ with $w^{\mathbf{K}} \in \mathring{W}^{\mathbf{K}}_{\mathbf{J}}$ and $w_{\mathbf{K}} \in \mathring{W}_{\mathbf{K}}$, where $w^{\mathbf{K}}$ is a unique element of the smallest length in the right coset $w\mathring{W}_{\mathbf{K}}$. For each $\mathbf{K} \subset \mathbf{J}$, we set

$$\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}} := \mathring{\Delta}_{\mathbf{J}} \setminus \mathring{\Delta}_{\mathbf{K}}, \qquad \mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}\pm} := \mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}} \cap \mathring{\Delta}_{\pm}.$$

In addition, for each $w \in \mathring{W}_{\mathbf{J}}$, we set

$$\Delta^{\mathbf{K}}_{\mathbf{J}}(w, \pm) := \{\, \alpha \in \Delta^{re}_{+} \mid \overline{\alpha} \in w\mathring{\Delta}^{\mathbf{K}}_{\mathbf{J}\pm} \,\}.$$

We denote it simply by $\Delta_{\mathbf{J}}(w, \pm)$ if $\mathbf{K} = \emptyset$, by $\Delta^{\mathbf{K}}(w, \pm)$ if $\mathbf{J} = \mathring{\mathbf{I}}$, and by $\Delta(w, \pm)$ if both $\mathbf{K} = \emptyset$ and $\mathbf{J} = \mathring{\mathbf{I}}$. Note that

$$\Delta_{\mathbf{J}+} = \Delta_{\mathbf{J}}(w, -) \amalg \Delta^{im}_{+} \amalg \Delta_{\mathbf{J}}(w, +), \quad \Delta_{\mathbf{J}}(w, +) = \Delta_{\mathbf{J}}(ww_\circ, -), \tag{2.1}$$

where $w_\circ$ is the longest element of $\mathring{W}_{\mathbf{J}}$.

Definition 2.3. We define $\widetilde{\boldsymbol{\mathcal{P}}}_{\mathbf{J}}$ and $\boldsymbol{\mathcal{P}}_{\mathbf{J}}$ to be the following sets:

$$\widetilde{\boldsymbol{\mathcal{P}}}_{\mathbf{J}} := \{(\mathbf{K}, u, y) \,|\, \mathbf{K} \subset \mathbf{J},\, u \in \mathring{W}^{\mathbf{K}}_{\mathbf{J}},\ y \in W_{\mathbf{K}} \,\}, \quad \boldsymbol{\mathcal{P}}_{\mathbf{J}} := \{(\mathbf{K}, u, y) \in \widetilde{\boldsymbol{\mathcal{P}}}_{\mathbf{J}} \,|\, \mathbf{K} \subsetneq \mathbf{J} \,\}.$$

For each $y \in \widehat{W}_{\mathbf{J}}$, we set

$$\Phi_{\mathbf{J}}(y) := \{\, \alpha \in \Delta_{\mathbf{J}+} \mid y^{-1}(\alpha) \in \Delta_{\mathbf{J}-} \,\}.$$

For each $(\mathbf{K}, u, y) \in \widetilde{\boldsymbol{\mathcal{P}}}_{\mathbf{J}}$, we define a subset $\nabla_{\mathbf{J}}(\mathbf{K}, u, y) \subset \Delta^{re}_{\mathbf{J}+}$ by setting

$$\nabla_{\mathbf{J}}(\mathbf{K}, u, y) := \Delta^{\mathbf{K}}_{\mathbf{J}}(u, -) \amalg u\Phi_{\mathbf{K}}(y).$$

Note that $\nabla_{\mathbf{J}}(\mathbf{K}, u, y) = \Phi_{\mathbf{J}}(y)$ if $\mathbf{K} = \mathbf{J}$ and that $\nabla_{\mathbf{J}}(\mathbf{K}, u, y) = \Delta_{\mathbf{J}}(u, -)$ if $\mathbf{K} = \emptyset$.

We call a subset $B \subset \Delta_{\mathbf{J}+}$ a *biconvex set* in $\Delta_{\mathbf{J}+}$ if it satisfies the following conditions:

$$\text{C(i)} \quad \beta,\gamma \in B,\ \ \beta+\gamma \in \Delta_{\mathbf{J}+} \Longrightarrow \beta+\gamma \in B\,;$$
$$\text{C(ii)} \quad \beta,\gamma \in \Delta_{\mathbf{J}+}\setminus B,\ \ \beta+\gamma \in \Delta_{\mathbf{J}+} \Longrightarrow \beta+\gamma \in \Delta_{\mathbf{J}+}\setminus B.$$

If, in addition, B is a subset of the set $\Delta^{re}_{\mathbf{J}+}$, then B is called a *real biconvex set* in $\Delta_{\mathbf{J}+}$. Let $\mathfrak{B}_{\mathbf{J}}$ be the set of all finite biconvex sets in $\Delta_{\mathbf{J}+}$, and $\mathfrak{B}^{\infty}_{\mathbf{J}}$ to be the set of all infinite real biconvex sets in $\Delta_{\mathbf{J}+}$. We set $\mathfrak{B}^{*}_{\mathbf{J}} := \mathfrak{B}_{\mathbf{J}} \amalg \mathfrak{B}^{\infty}_{\mathbf{J}}$.

In the case where $\mathbf{J} = \mathring{\mathbf{I}}$, we will denote the symbols above more simply by removing $\mathbf{J}$ from them.

Theorem 2.4 ([7]). *For each non-empty subset $\mathbf{J} \subset \mathring{\mathbf{I}}$, the assignment $(\kappa, u, y) \mapsto \nabla_{\mathbf{J}}(\kappa, u, y)$ defines a bijective mapping from $\widetilde{\boldsymbol{\mathcal{P}}}_{\mathbf{J}}$ to $\mathfrak{B}^{*}_{\mathbf{J}}$, which maps $\boldsymbol{\mathcal{P}}_{\mathbf{J}}$ onto $\mathfrak{B}^{\infty}_{\mathbf{J}}$.*

For each non-empty subset $\mathbf{J} \subset \mathring{\mathbf{I}}$, we set

$$\mathrm{Aut}(\Delta_{\mathbf{J}}) := \{\, \phi \in \mathrm{O}(\mathfrak{h}'^{*}_{\mathbf{J}}) \mid \phi(\Delta_{\mathbf{J}}) = \Delta_{\mathbf{J}} \,\},$$
$$\mathrm{Aut}(\Delta_{\mathbf{J}}, \Pi_{\mathbf{J}}) := \{\, \phi \in \mathrm{Aut}(\Delta_{\mathbf{J}}) \mid \phi(\Pi_{\mathbf{J}}) = \Pi_{\mathbf{J}} \,\}.$$

For each $\mathbf{K} \subset \mathbf{J}$, we set

$$\mathrm{Aut}(\Delta_{\mathbf{J}})^{\mathbf{K}} := \{\, \phi \in \mathrm{Aut}(\Delta_{\mathbf{J}}) \mid \phi(\mathring{\Pi}_{\mathbf{K}}) \subset \Delta_{\mathbf{J}+} \,\},$$
$$W^{\mathbf{K}}_{\mathbf{J}} := W_{\mathbf{J}} \cap \mathrm{Aut}(\Delta_{\mathbf{J}})^{\mathbf{K}}, \qquad \widehat{W}^{\mathbf{K}}_{\mathbf{J}} := \widehat{W}_{\mathbf{J}} \cap \mathrm{Aut}(\Delta_{\mathbf{J}})^{\mathbf{K}}.$$

Note that $\mathrm{Aut}(\Delta_{\mathbf{J}}, \Pi_{\mathbf{J}}) \subset \mathrm{Aut}(\Delta_{\mathbf{J}})^{\mathbf{J}}$. In addition, we set

$$\Omega_{\mathbf{J}} := \widehat{W}_{\mathbf{J}} \cap \mathrm{Aut}(\Delta_{\mathbf{J}}, \Pi_{\mathbf{J}}).$$

Then

$$\widehat{W}_{\mathbf{J}} = W_{\mathbf{J}} \rtimes \Omega_{\mathbf{J}} \subset \mathrm{Aut}(\Delta_{\mathbf{J}}). \tag{2.2}$$

Let $\ell_{\mathbf{J}} \colon \widehat{W}_{\mathbf{J}} \to \mathbb{Z}_{+}$ be the extended length function defined by setting $\ell_{\mathbf{J}}(x\rho) := \ell_{\mathbf{J}}(x)$ for each $x \in W_{\mathbf{J}}$ and $\rho \in \Omega_{\mathbf{J}}$. We note that $\ell_{\mathbf{J}}(y) = \varPhi_{\mathbf{J}}(y)$ for all $y \in \widehat{W}_{\mathbf{J}}$.

Proposition 2.5 ([9]). *For each connected subset $\mathbf{J} \subset \mathring{\mathbf{I}}$, the assignment*

$$j \mapsto \rho_{\mathbf{J}j} := t_{\varepsilon_j} w_{\circ j} w_{\circ} \tag{2.3}$$

defines a bijective mapping from the set $\mathbf{J}_{*} := \{j \in \mathbf{J} \mid (\varepsilon_j \mid \theta_{\mathbf{J}}) = 1\}$ to $\Omega_{\mathbf{J}} \setminus \{1\}$. Here, $w_{\circ}$ and $w_{\circ j}$ are the longest elements of $\mathring{W}_{\mathbf{J}}$ and $\mathring{W}_{\mathbf{J}\setminus\{j\}}$, respectively. Moreover, the condition that $\rho(\delta - \theta_{\mathbf{J}}) = \alpha_j$ for $\rho \in \Omega_{\mathbf{J}} \setminus \{1\}$ and $j \in \mathbf{J}$ is equivalent to the condition that $\rho = \rho_{\mathbf{J}j}$ with $j \in \mathbf{J}_{*}$.

Proof. Although the setting of Proposition 1.18 in [9] is different from that of this case, the proof can be applied to this case by modifying suitably. □

Lemma 2.6. *Let $\mathbf{J}$ be an arbitrary connected subset of $\mathring{\mathbf{I}}$, and $\mathbf{K}$ an arbitrary subset of $\mathbf{J}$. Then each $\phi \in \mathrm{Aut}(\Delta_{\mathbf{J}})$ can be uniquely written as $\phi = \phi^{\mathbf{K}}\phi_{\mathbf{K}}$ with $\phi^{\mathbf{K}} \in \mathrm{Aut}(\Delta_{\mathbf{J}})^{\mathbf{K}}$ and $\phi_{\mathbf{K}} \in \mathring{W}_{\mathbf{K}}$.*

Proof. We first prove the uniqueness. Suppose that $\phi = a^{\mathbf{K}}a_{\mathbf{K}} = b^{\mathbf{K}}b_{\mathbf{K}}$ with $a^{\mathbf{K}}, b^{\mathbf{K}} \in \mathrm{Aut}(\Delta_{\mathbf{J}})^{\mathbf{K}}$ and $a_{\mathbf{K}}, b_{\mathbf{K}} \in \mathring{W}_{\mathbf{K}}$. Then $a^{\mathbf{K}} = b^{\mathbf{K}}b_{\mathbf{K}}a^{-1}_{\mathbf{K}}$. Since $a^{\mathbf{K}} \in \mathrm{Aut}(\Delta_{\mathbf{J}})^{\mathbf{K}}$ and $b_{\mathbf{K}}a^{-1}_{\mathbf{K}} \in \mathring{W}_{\mathbf{J}}$, we have $b_{\mathbf{K}}a^{-1}_{\mathbf{K}} = 1$, hence $b_{\mathbf{K}} = a_{\mathbf{K}}$ and $b^{\mathbf{K}} = a^{\mathbf{K}}$. We next prove the existence. By Corollary 3.10 in [11], the ϕ can be uniquely written as $\sigma\rho z$ with $\sigma = \pm 1$, $\rho \in \mathrm{Aut}(\Delta_{\mathbf{J}}, \Pi_{\mathbf{J}})$, and $z \in W_{\mathbf{J}}$. Moreover, we see that z can be uniquely written as xy with $x \in W_{\mathbf{J}}^{\mathbf{K}}$ and $y \in \mathring{W}_{\mathbf{K}}$. Hence $\phi = \sigma\rho xy$. In the case where $\sigma = 1$, put $\phi^{\mathbf{K}} = \rho x$ and $\phi_{\mathbf{K}} = y$. Then $\phi = \phi^{\mathbf{K}}\phi_{\mathbf{K}}$ with $\phi^{\mathbf{K}} \in \mathrm{Aut}(\Delta_{\mathbf{J}})^{\mathbf{K}}$ and $\phi_{\mathbf{K}} \in \mathring{W}_{\mathbf{K}}$. In the case where $\sigma = -1$, put $\phi^{\mathbf{K}} = -\rho x w_\circ$ and $\phi_{\mathbf{K}} = w_\circ y$, where $w_\circ$ is the longest element of $\mathring{W}_{\mathbf{K}}$. Then $\phi = \phi^{\mathbf{K}}\phi_{\mathbf{K}}$ with $\phi^{\mathbf{K}} \in \mathrm{Aut}(\Delta_{\mathbf{J}})^{\mathbf{K}}$ and $\phi_{\mathbf{K}} \in \mathring{W}_{\mathbf{K}}$. □

Lemma 2.7. *Let* $\mathbf{J}$ *and* $\mathbf{J}'$ *be connected subsets of* $\mathring{\mathbf{I}}$ *which are disjointed with each other.*

(1) *For each* $j \in \mathbf{J}_*$, *there exists a unique element* $w_{\mathbf{J}j} \in \mathring{W}_{\mathbf{J}}$ *such that*

$$t_{\varepsilon_j}|_{\mathfrak{h}'^*_{\mathbf{J}}} = \rho_{\mathbf{J}j} w_{\mathbf{J}j}. \tag{2.4}$$

Moreover, the following equalities hold:

$$\text{(i)}\ \ \rho_{\mathbf{J}j} = (t_{\varepsilon_j})^{\mathbf{J}}|_{\mathfrak{h}'^*_{\mathbf{J}}}, \qquad \text{(ii)}\ \ w_{\mathbf{J}j} = (t_{\varepsilon_j})_{\mathbf{J}} = w_\circ w_{\circ j}. \tag{2.5}$$

Here, $(t_{\varepsilon_j})^{\mathbf{J}} \in \widehat{W}^{\mathbf{J}}$ *and* $(t_{\varepsilon_j})_{\mathbf{J}} \in \mathring{W}_{\mathbf{J}}$ *are unique elements such that* $t_{\varepsilon_j} = (t_{\varepsilon_j})^{\mathbf{J}}(t_{\varepsilon_j})_{\mathbf{J}}$, *and* $w_\circ$ *and* $w_{\circ j}$ *are the longest elements of* $\mathring{W}_{\mathbf{J}}$ *and* $\mathring{W}_{\mathbf{J}\setminus\{j\}}$ *respectively.*

(2) *For each* $i \in \mathring{\mathbf{I}} \setminus \mathbf{J}$, $z \in \mathring{W}_{\mathbf{J}'}$, *and* $j' \in \mathbf{J}'_*$, *the following equalities hold*:

$$\text{(i)}\ [(t_{\varepsilon_j})^{\mathbf{J}}, t_{\varepsilon_i}] = 0, \quad \text{(ii)}\ [(t_{\varepsilon_j})^{\mathbf{J}}, z] = 0, \quad \text{(iii)}\ [(t_{\varepsilon_j})^{\mathbf{J}}, (t_{\varepsilon_{j'}})^{\mathbf{J}'}] = 0. \tag{2.6}$$

Here, [,] *is the commutator, i.e.,* $[a, b] = ab - ba$. *Moreover,*

$$\ell((t_{\varepsilon_j})^{\mathbf{J}} t_{\varepsilon_i}) = \ell((t_{\varepsilon_j})^{\mathbf{J}}) + \ell(t_{\varepsilon_i}), \tag{2.7}$$

$$\ell(t_{\varepsilon_j})^{\mathbf{J}} z) = \ell(t_{\varepsilon_j})^{\mathbf{J}}) + \ell(z), \tag{2.8}$$

$$\ell((t_{\varepsilon_j})^{\mathbf{J}}(t_{\varepsilon_{j'}})^{\mathbf{J}'}) = \ell((t_{\varepsilon_j})^{\mathbf{J}}) + \ell((t_{\varepsilon_{j'}})^{\mathbf{J}'}). \tag{2.9}$$

(3) *For each* $j \in \mathbf{J}_*$ *and* $\beta \in \Delta_{\mathbf{J}'}$, *the equality* $(t_{\varepsilon_j})^{\mathbf{J}}(\beta) = \beta$ *holds.*

(4) *For each* $j \in \mathbf{J}_*$, *the element* $(t_{\varepsilon_j})^{\mathbf{J}}$ *satisfies the following equalities*:

$$\Phi((t_{\varepsilon_j})^{\mathbf{J}}) \subset \Delta^{\mathbf{J}}(1, -), \tag{2.10}$$

$$\Phi(t_{\varepsilon_j}) \cap \Delta_{\mathbf{J}+} = (t_{\varepsilon_j})^{\mathbf{J}} \Phi(w_{\mathbf{J}_j}), \tag{2.11}$$

$$(t_{\varepsilon_j})^{\mathbf{J}} \Delta^{\mathbf{J}}(1, -) \subset \Delta^{\mathbf{J}}(1, -). \tag{2.12}$$

Moreover, $\ell((t_{\varepsilon_j})^{\mathbf{J}}) = 0$ *if and only if* $\mathbf{J} = \mathring{\mathbf{I}}$.

(5) *For each* $j \in \mathbf{J}_*$, *there exists a unique element* $j^- \in \mathbf{J}_*$ *such that*

$$\text{(i)}\ \ \rho_{\mathbf{J}j}(\alpha_{j^-}) = \delta - \theta_{\mathbf{J}}, \quad \text{(ii)}\ \ \rho_{\mathbf{J}j^-} = (\rho_{\mathbf{J}j})^{-1}. \tag{2.13}$$

In addition, $(\rho_{\mathbf{J}j})^2 = 1$ *if and only if* $j^- = j$. *Moreover, the following equalities hold*:

$$(t_{\varepsilon_j})^{\mathbf{J}} s_{j^-} (t_{\varepsilon_{j^-}})^{\mathbf{J}}|_{\mathfrak{h}'^*_{\mathbf{J}}} = s_{\delta - \theta_{\mathbf{J}}}, \tag{2.14}$$

$$\Phi((t_{\varepsilon_j})^{\mathbf{J}} s_{j^-} (t_{\varepsilon_{j^-}})^{\mathbf{J}}) \cap \Delta_{\mathbf{J}+} = \{\delta - \theta_{\mathbf{J}}\}, \tag{2.15}$$

$$\Phi((t_{\varepsilon_j})^{\mathbf{J}} s_{j^-} (t_{\varepsilon_{j^-}})^{\mathbf{J}}) \setminus \{\delta - \theta_{\mathbf{J}}\} \subset \Delta^{\mathbf{J}}(1, -), \tag{2.16}$$

$$\ell((t_{\varepsilon_j})^{\mathbf{J}} s_{j^-} (t_{\varepsilon_{j^-}})^{\mathbf{J}}) = \ell((t_{\varepsilon_j})^{\mathbf{J}}) + 1 + \ell((t_{\varepsilon_{j^-}})^{\mathbf{J}}). \tag{2.17}$$

Proof. Let us prove the part (1). Set $w_{\mathbf{J}j} := w_\circ w_{\circ j}$. Then $w_{\mathbf{J}j} \in \overset{\circ}{W}_{\mathbf{J}}$. By Proposition 2.5, we have $t_{\varepsilon_j}|_{\mathfrak{h}'^*_{\mathbf{J}}} = \rho_{\mathbf{J}j} w_{\mathbf{J}j}$. On the other hand, by Lemma 2.6, we have $t_{\varepsilon_j} = (t_{\varepsilon_j})^{\mathbf{J}}(t_{\varepsilon_j})_{\mathbf{J}}$ with $(t_{\varepsilon_j})^{\mathbf{J}} \in \widehat{W}^{\mathbf{J}}$ and $(t_{\varepsilon_j})_{\mathbf{J}} \in \overset{\circ}{W}_{\mathbf{J}}$. It follows that $t_{\varepsilon_j}|_{\mathfrak{h}'^*_{\mathbf{J}}} = (t_{\varepsilon_j})^{\mathbf{J}}|_{\mathfrak{h}'^*_{\mathbf{J}}}(t_{\varepsilon_j})_{\mathbf{J}}$. Hence (2.5) follows from Lemma 2.6. The uniqueness of the decomposition (2.4) follows from (2.5).

Let us prove the part (2). By the part (1), we have $(t_{\varepsilon_j})^{\mathbf{J}} = t_{\varepsilon_j} w_{\mathbf{J}j}^{-1}$. It is clear that $[t_{\varepsilon_j}, t_{\varepsilon_i}] = [w_{\mathbf{J}j}^{-1}, t_{\varepsilon_i}] = 0$, which implies (2.6)(i). Since $(\varepsilon_j \,|\, \alpha) = 0$ for all $\alpha \in \overset{\circ}{\Pi}_{\mathbf{J}'}$, we have $[t_{\varepsilon_j}, s_\alpha] = 0$. Since $w_{\mathbf{J}j} \in \overset{\circ}{W}_{\mathbf{J}}$, we see that $[w_{\mathbf{J}j}^{-1}, s_\alpha] = 0$ for all $\alpha \in \overset{\circ}{\Pi}_{\mathbf{J}'}$. Thus we get $[(t_{\varepsilon_j})^{\mathbf{J}}, s_\alpha] = 0$ for all $\alpha \in \overset{\circ}{\Pi}_{\mathbf{J}'}$, which implies (2.6)(ii). Since $(t_{\varepsilon_{j'}})^{\mathbf{J}'} = t_{\varepsilon_{j'}} w_{\mathbf{J}'j'}^{-1}$ with $w_{\mathbf{J}'j'} \in \overset{\circ}{W}_{\mathbf{J}'}$, (iii) follows from (i) and (ii).

It is clear that $\ell((t_{\varepsilon_j})^{\mathbf{J}} t_{\varepsilon_i}) \leq \ell((t_{\varepsilon_j})^{\mathbf{J}}) + \ell(t_{\varepsilon_i})$. Since $[w_{\mathbf{J}j}, t_{\varepsilon_i}] = 0$ we have $t_{\varepsilon_j} t_{\varepsilon_i} = (t_{\varepsilon_j})^{\mathbf{J}} t_{\varepsilon_i} w_{\mathbf{J}j}$, and hence

$$\ell(t_{\varepsilon_j} t_{\varepsilon_i}) \leq \ell((t_{\varepsilon_j})^{\mathbf{J}} t_{\varepsilon_i}) + \ell(w_{\mathbf{J}j}).$$

On the other hand, we have

$$\ell(t_{\varepsilon_j} t_{\varepsilon_i}) = \ell(t_{\varepsilon_j}) + \ell(t_{\varepsilon_i}) = \{\ell((t_{\varepsilon_j})^{\mathbf{J}}) + \ell(w_{\mathbf{J}j})\} + \ell(t_{\varepsilon_i}).$$

Thus we get $\ell((t_{\varepsilon_j})^{\mathbf{J}}) + \ell(t_{\varepsilon_i}) \leq \ell((t_{\varepsilon_j})^{\mathbf{J}} t_{\varepsilon_i})$, which implies (2.7). The (2.8) is clear. It is easy to see that

$$\ell((t_{\varepsilon_j})^{\mathbf{J}} t_{\varepsilon_{j'}}) \leq \ell((t_{\varepsilon_j})^{\mathbf{J}} (t_{\varepsilon_{j'}})^{\mathbf{J}'}) + \ell(w_{\mathbf{J}'j'}).$$

From (2.7) and (2.8), it follows that

$$\ell((t_{\varepsilon_j})^{\mathbf{J}} t_{\varepsilon_{j'}}) = \ell((t_{\varepsilon_j})^{\mathbf{J}}) + \ell((t_{\varepsilon_{j'}})^{\mathbf{J}'}) + \ell(w_{\mathbf{J}'j'}).$$

Thus we get that $\ell((t_{\varepsilon_j})^{\mathbf{J}}) + \ell((t_{\varepsilon_{j'}})^{\mathbf{J}'}) \leq \ell((t_{\varepsilon_j})^{\mathbf{J}} (t_{\varepsilon_{j'}})^{\mathbf{J}'})$, which implies (2.9), since $\ell((t_{\varepsilon_j})^{\mathbf{J}} (t_{\varepsilon_{j'}})^{\mathbf{J}'}) \leq \ell((t_{\varepsilon_j})^{\mathbf{J}}) + \ell((t_{\varepsilon_{j'}})^{\mathbf{J}'})$.

Let us prove the part (3). Since $t_{\varepsilon_j}(\beta) = \beta$ and $w_{\mathbf{J}j}(\beta) = \beta$, we have $(t_{\varepsilon_j})^{\mathbf{J}}(\beta) = t_{\varepsilon_j} w_{\mathbf{J}j}^{-1}(\beta) = \beta$.

Let us prove the part (4). It is easy to see that

$$\Phi(t_{\varepsilon_j}) = \Phi((t_{\varepsilon_j})^{\mathbf{J}}) \amalg (t_{\varepsilon_j})^{\mathbf{J}} \Phi(w_{\mathbf{J}j}) \subset \Delta(1,-). \tag{2.18}$$

By (2.7), we have $\Phi((t_{\varepsilon_j})^{\mathbf{J}}) \cap \Delta_{\mathbf{J}+} = \emptyset$, hence we get (2.10) and (2.11) follow by (2.18). Since both t_{ε_j} and $w_{\mathbf{J}j}^{-1}$ stabilize $\Delta^{\mathbf{J}}(1,-)$, the product $(t_{\varepsilon_j})^{\mathbf{J}} = t_{\varepsilon_j} w_{\mathbf{J}j}^{-1}$ stabilizes $\Delta^{\mathbf{J}}(1,-)$. We see that $\Phi(t_{\varepsilon_j}) \cap \Delta^{\mathbf{J}}(1,-) = \emptyset$ if and only if $\mathbf{J} = \overset{\circ}{\mathbf{I}}$, hence the second assertion follows from (2.11) and (2.18).

Let us prove the part (5). By Proposition 2.5, there exists a unique element $j^- \in \mathbf{J}_*$ satisfying (2.13). Suppose that $(\rho_{\mathbf{J}j})^2 = 1$, i.e., $(\rho_{\mathbf{J}j})^{-1} = \rho_{\mathbf{J}j}$. By (2.13)(ii) and Proposition 2.5 we get $j^- = j$. Suppose that $j^- = j$. Then, by (2.13)(ii) we get $\rho_{\mathbf{J}j} = (\rho_{\mathbf{J}j})^{-1}$, i.e., $(\rho_{\mathbf{J}j})^2 = 1$. By (2.5)(i) and (2.13)(ii), we see that

$$(t_{\varepsilon_j})^{\mathbf{J}} s_{j^-} (t_{\varepsilon_{j^-}})^{\mathbf{J}}|_{\mathfrak{h}'^*_{\mathbf{J}}} = \rho_{\mathbf{J}j} s_{j^-} (\rho_{\mathbf{J}j})^{-1}. \tag{2.19}$$

Since $\delta - \theta_{\mathbf{J}} = \rho_{\mathbf{J}j}(\alpha_{j^-})$, we have

$$\rho_{\mathbf{J}j} s_{j^-} (\rho_{\mathbf{J}j})^{-1}(\delta - \theta_{\mathbf{J}}) = -(\delta - \theta_{\mathbf{J}}). \tag{2.20}$$

Since $(\alpha_{j^-} \,|\, \alpha_{j^-}) = (\delta - \theta_{\mathbf{J}} \,|\, \delta - \theta_{\mathbf{J}})$ and $((\rho_{\mathbf{J}j})^{-1}(\alpha_i) \,|\, \alpha_{j^-}) = (\alpha_i \,|\, \delta - \theta_{\mathbf{J}})$ for all $i \in \mathbf{J}$, we have

$$s_{j^-}(\rho_{\mathbf{J}j})^{-1}(\alpha_i) = (\rho_{\mathbf{J}j})^{-1}(\alpha_i) - \tfrac{2(\alpha_i \,|\, \delta-\theta_{\mathbf{J}})}{(\delta-\theta_{\mathbf{J}} \,|\, \delta-\theta_{\mathbf{J}})}\alpha_{j^-},$$

which implies

$$\rho_{\mathbf{J}j}s_{j^-}(\rho_{\mathbf{J}j})^{-1}(\alpha_i) = \alpha_i - \tfrac{2(\alpha_i \,|\, \delta-\theta_{\mathbf{J}})}{(\delta-\theta_{\mathbf{J}} \,|\, \delta-\theta_{\mathbf{J}})}(\delta - \theta_{\mathbf{J}}) = s_{\delta-\theta_{\mathbf{J}}}(\alpha_i). \tag{2.21}$$

Therefore (2.14) follows from (2.19)(2.20)(2.21).

By (2.10), we see that

$$s_{j^-}\Phi((t_{\varepsilon_{j^-}})^{\mathbf{J}}) \subset \Delta^{\mathbf{J}}(1,-), \tag{2.22}$$

$$\Phi(s_{j^-}(t_{\varepsilon_{j^-}})^{\mathbf{J}}) = \{\alpha_{j^-}\} \amalg s_{j^-}\Phi((t_{\varepsilon_{j^-}})^{\mathbf{J}}), \tag{2.23}$$

since $s_{j^-}\Delta^{\mathbf{J}}(1,-) \subset \Delta^{\mathbf{J}}(1,-)$. By (2.12) and (2.22), we have

$$(t_{\varepsilon_j})^{\mathbf{J}} s_{j^-}\Phi((t_{\varepsilon_{j^-}})^{\mathbf{J}}) \subset \Delta^{\mathbf{J}}(1,-). \tag{2.24}$$

By (2.23)(2.24) and the equality $(t_{\varepsilon_j})^{\mathbf{J}}(\alpha_{j^-}) = \delta - \theta_{\mathbf{J}}$, we have

$$\Phi((t_{\varepsilon_j})^{\mathbf{J}} s_{j^-}(t_{\varepsilon_{j^-}})^{\mathbf{J}}) = \Phi((t_{\varepsilon_j})^{\mathbf{J}}) \amalg \{\delta - \theta_{\mathbf{J}}\} \amalg (t_{\varepsilon_j})^{\mathbf{J}} s_{j^-}\Phi((t_{\varepsilon_{j^-}})^{\mathbf{J}}). \tag{2.25}$$

Therefore (2.15), (2.16), and (2.17) follow from (2.10)(2.24)(2.25). □

Lemma 2.8. *Let us use the notations as in Proposition 2.5. Assume that* $\mathbf{J}$ *is a connected subset of* $\mathring{\mathbf{I}}$ *with* $\sharp\mathbf{J} \geq 2$ *and that an element* $j \in \mathbf{J}_*$ *satisfies* $(\rho_{\mathbf{J}j})^2 = 1$. *Then there exist distinct elements* $i, i' \in \mathbf{I}$ *and an element* $z \in W$ *satisfying* $\Phi(z) \subset \Delta(1,-)$, $\alpha_j = z(\alpha_i)$, *and* $\delta - \theta_{\mathbf{J}} = z(\alpha_{i'})$.

Proof. Let B be the subset of $\Delta(1,-)$ consisting of all β such that

$$\beta + \alpha_{i_1} + \cdots + \alpha_{i_n} = \delta - \theta_{\mathbf{J}} \tag{2.26}$$

for some sequence $(i_1, \ldots, i_n)$ consisting of elements of $\mathring{\mathbf{I}}$ with $n \in \mathbb{N}$. Then, it is easy to see that both B and $B' = B \amalg \{\delta - \theta_{\mathbf{J}}\}$ are finite biconvex sets. Hence, there exist unique $z \in W$ and $i' \in \mathbf{I}$ such that $B = \Phi(z)$ and $\delta - \theta_{\mathbf{J}} = z(\alpha_{i'})$ by Theorem 2.4. We next show that

$$s_j(\delta - \theta_{\mathbf{J}}) = \delta - \theta_{\mathbf{J}}, \tag{2.27}$$

$$s_j(B) = B. \tag{2.28}$$

By the assumption of the Lemma and the extended Dynkin diagram of $\mathring{\Delta}_{\mathbf{J}}$, we see that $(\delta - \theta_{\mathbf{J}} \,|\, \alpha_j) = 0$, which implies (2.27). Let β be an arbitrary element of B. To prove (2.28), it suffices to show that B includes the α_j-string through β. Since α_j is not a short root, we see that the length of the α_j-string through β is less than 2. If the length is 1, there is nothing to prove. Suppose that the length is 2. In the case where $s_j(\beta) = \beta - \alpha_j$, we see that $s_j(\beta) \in \Delta(1,-)$ and

$$s_j(\beta) + \alpha_j + (\alpha_{i_1} + \cdots + \alpha_{i_n}) = \delta - \theta_{\mathbf{J}},$$

which implies $s_j(\beta) \in B$. In the case where $s_j(\beta) = \beta + \alpha_j$, we see that $j = i_k$ for some $1 \leq k \leq n$. Indeed, if $j \neq i_k$ for all $1 \leq k \leq n$, then we see that $\beta + (\alpha_{i_1} + \cdots + \alpha_{i_n}) + m\alpha_j = \delta - \theta_{\mathbf{J}}$ for some $m \geq 1$ by applying s_j to the equality (2.26). Here we use (2.27). This contradicts to (2.26). Hence, $j = i_k$ for some $1 \leq k \leq n$. Thus we see that

$$s_j(\beta) + (\alpha_{i_1} + \cdots + \alpha_{i_{k-1}} + \alpha_{i_{k+1}} + \cdots + \alpha_{i_n}) = \delta - \theta_{\mathbf{J}}$$

with $s_j(\beta) \in \Delta(1,-)$. Here we have $n \geq 2$. Indeed, if $n = 1$ then $s_j(\beta) = \delta - \theta_{\mathbf{J}}$, which contradicts to (2.27). Thus we get $s_j(\beta) \in B$.

By (2.28) and the equality $B = \Phi(z)$, we have

$$\Phi(s_j z) = \{\alpha_j\} \amalg s_j\Phi(z) = \{\alpha_j\} \amalg \Phi(z). \tag{2.29}$$

Since $\Phi(z) \subset \Phi(s_j z)$ and $\sharp\{\Phi(s_j z) \setminus \Phi(z)\} = 1$, we see that

$$\Phi(s_j z) = \Phi(z) \amalg z\{\alpha_i\} = \Phi(z s_i). \tag{2.30}$$

for some unique $i \in \mathbf{I}$. By (2.29)(2.30), we get $\alpha_j = z(\alpha_i)$. □

3. Notations and preliminary results for reduced words and convex orders

Throughout this section, we assume that $\mathfrak{g}$ is the affine Kac-Moody Lie algebra of the type $X_r^{(1)}$ $(X = A, B, C, D, E, F, G)$ with Δ the root system.

We denote by $\mathbb{N}_n$ the set $\{m \in \mathbb{N} \,|\, m \leq n\}$ for each $n \in \mathbb{N}$, and set $\mathbb{N}_\infty := \mathbb{N}$ and $\mathbb{N}_* := \mathbb{N} \amalg \{\infty\}$, where ∞ is a symbol. We extend the usual order $\leq$ on $\mathbb{N}$ to a total order on $\mathbb{N}_*$ by setting $n < \infty$ for each $n \in \mathbb{N}$. We also set $\infty + n = n + \infty = \infty n = n\infty = \infty$ for each $n \in \mathbb{N}_*$.

For each non-empty subset $\mathbf{J}$ of $\mathring{\mathbf{I}}$, we set

$$\widehat{S}_{\mathbf{J}} := S_{\mathbf{J}} \amalg (\Omega_{\mathbf{J}} \setminus \{1\}) = \coprod_{c=1}^{\mathrm{C}(\mathbf{J})} (S_{\mathbf{J}_c} \amalg \Omega_{\mathbf{J}_c} \setminus \{1\}).$$

For each $n \in \mathbb{N}_*$, we denote a sequence consisting of elements $\boldsymbol{s}(p) \in \widehat{S}_{\mathbf{J}}$ with $p \in \mathbb{N}_n$ by $\boldsymbol{s} = (\boldsymbol{s}(p))_{p \in \mathbb{N}_n}$, and denote the set of such all sequences by $\widehat{S}_{\mathbf{J}}^{\mathbb{N}_n}$. For each $\boldsymbol{s} \in \widehat{S}_{\mathbf{J}}^{\mathbb{N}_n}$ and $m \in \mathbb{N}_n$, we define a sequence $\boldsymbol{s}_{|m} \in \widehat{S}_{\mathbf{J}}^{\mathbb{N}_m}$ by setting $\boldsymbol{s}_{|m}(p) := \boldsymbol{s}(p)$ for each $p \in \mathbb{N}_m$, and call the sequence $\boldsymbol{s}_{|m}$ the initial m-section of $\boldsymbol{s}$. Let $\{\boldsymbol{s}_p\}_{p \in \mathbb{N}}$ be a family of finite sequences of elements of $\widehat{S}_{\mathbf{J}}$ such that $\boldsymbol{s}_p$ is the initial m_p-section of $\boldsymbol{s}_{p+1}$ with $m_p < m_{p+1}$ for each $p \in \mathbb{N}$. Then we see that there exists a unique infinite sequence $\boldsymbol{s}_\infty$ of elements of $\widehat{S}_{\mathbf{J}}$ such that $\boldsymbol{s}_p$ is the initial m_p-section of $\boldsymbol{s}_\infty$ for each $p \in \mathbb{N}$, and denote by $\lim_{p \to \infty} \boldsymbol{s}_p$ the infinite sequence $\boldsymbol{s}_\infty$. For each $\boldsymbol{s} \in \widehat{S}_{\mathbf{J}}^{\mathbb{N}_n}$ and $\boldsymbol{s}' \in \widehat{S}_{\mathbf{J}}^{\mathbb{N}_{n'}}$ with $n < \infty$ and $n' \in \mathbb{N}_*$, we define a sequence $\boldsymbol{s}\boldsymbol{s}' = (\boldsymbol{s}\boldsymbol{s}'(p))_{p \in \mathbb{N}_{n+n'}} \in \widehat{S}_{\mathbf{J}}^{\mathbb{N}_{n+n'}}$ by setting

$$\boldsymbol{s}\boldsymbol{s}'(p) := \boldsymbol{s}(p) \quad \text{for } p \leq n, \qquad \boldsymbol{s}\boldsymbol{s}'(p) := \boldsymbol{s}'(p-n) \quad \text{for } n+1 \leq p.$$

The product $\boldsymbol{s}\boldsymbol{s}'$ satisfies the associative law: $(\boldsymbol{s}\boldsymbol{s}')\boldsymbol{s}'' = \boldsymbol{s}(\boldsymbol{s}'\boldsymbol{s}'')$ for $\boldsymbol{s} \in S^{\mathbb{N}_n}$, $\boldsymbol{s}' \in S^{\mathbb{N}_{n'}}$, $\boldsymbol{s}'' \in S^{\mathbb{N}_{n''}}$ with $n, n' < \infty$. Therefore, the product $\boldsymbol{s}_1 \cdots \boldsymbol{s}_{p-1}\boldsymbol{s}_p$ is defined naturally for each family $\{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_{p-1}, \boldsymbol{s}_p\}$ of sequences of elements of $\widehat{S}_{\mathbf{J}}$ such that $\boldsymbol{s}_i$ for $i \in \mathbb{N}_{p-1}$ are finite sequences.

For each $\boldsymbol{s} \in \widehat{S}_{\mathbf{J}}^{\mathbb{N}_n}$ with $n < \infty$, we define an element $[\boldsymbol{s}]$ of $\widehat{W}_{\mathbf{J}}$ by setting

$$[\boldsymbol{s}] := \boldsymbol{s}(1)\boldsymbol{s}(2)\cdots\boldsymbol{s}(n).$$

For each $n \in \mathbb{N}_*$, we call an element $\boldsymbol{s} \in \widehat{S}_{\mathbf{J}}^{\mathbb{N}_n}$ a *reduced word* of $(\widehat{W}_{\mathbf{J}}, \widehat{S}_{\mathbf{J}})$ if

$$\ell_{\mathbf{J}}([\boldsymbol{s}_{|p-1}]) \leq \ell_{\mathbf{J}}([\boldsymbol{s}_{|p}])$$

for all $p \in \mathbb{N}_n$. Here, $\ell_{\mathbf{J}} \colon \widehat{W}_{\mathbf{J}} \to \mathbb{Z}_+$ is the extended length function.

For each reduced word $\boldsymbol{s} = (\boldsymbol{s}(p))_{p \in \mathbb{N}_n}$ of $(\widehat{W}_{\mathbf{J}}, \widehat{S}_{\mathbf{J}})$ with $n \in \mathbb{N}_*$, we set

$$\boldsymbol{s}^{-1}(S_{\mathbf{J}}) := \{\, p \in \mathbb{N}_n \mid \boldsymbol{s}(p) \in S_{\mathbf{J}} \,\}, \qquad \ell_{\mathbf{J}}(\boldsymbol{s}) := \sharp\boldsymbol{s}^{-1}(S_{\mathbf{J}}),$$

and call the non-negative integer $\ell_{\mathbf{J}}(\boldsymbol{s})$ *the length of* $\boldsymbol{s}$. We denote by $\widehat{\mathcal{W}}_{\mathbf{J}}^{n}$ the set of all reduced words with length n and set $\widehat{\mathcal{W}}_{\mathbf{J}} := \amalg_{n\in\mathbb{N}}\widehat{\mathcal{W}}_{\mathbf{J}}^{n}$ and $\widehat{\mathcal{W}}_{\mathbf{J}}^{*} := \widehat{\mathcal{W}}_{\mathbf{J}} \amalg \widehat{\mathcal{W}}_{\mathbf{J}}^{\infty}$. We call an element of $\widehat{\mathcal{W}}_{\mathbf{J}}$ (resp. $\widehat{\mathcal{W}}_{\mathbf{J}}^{\infty}$) *a finite reduced word* (resp. *an infinite reduced word*) of $(\widehat{W}_{\mathbf{J}}, \widehat{S}_{\mathbf{J}})$. For each $n \in \mathbb{N}_{*}$, we denote by $\mathcal{W}_{\mathbf{J}}^{n}$ the subset of $\widehat{\mathcal{W}}_{\mathbf{J}}^{n}$ which consists of elements $\boldsymbol{s} \in \widehat{\mathcal{W}}_{\mathbf{J}}^{n}$ such that $\boldsymbol{s}(p) \in S_{\mathbf{J}}$ for all $p \in \mathbb{N}_{n}$, and call an element $\boldsymbol{s} \in \mathcal{W}_{\mathbf{J}}^{n}$ a reduced word of $(W_{\mathbf{J}}, S_{\mathbf{J}})$. We set $\mathcal{W}_{\mathbf{J}} := \amalg_{n\in\mathbb{N}}\mathcal{W}_{\mathbf{J}}^{n}$ and $\mathcal{W}_{\mathbf{J}}^{*} := \mathcal{W}_{\mathbf{J}} \amalg \mathcal{W}_{\mathbf{J}}^{\infty}$, and call an element of $\mathcal{W}_{\mathbf{J}}$ (resp. $\mathcal{W}_{\mathbf{J}}^{\infty}$) a finite reduced word (resp. an infinite reduced word) of $(W_{\mathbf{J}}, S_{\mathbf{J}})$.

For each reduced word $\boldsymbol{s} \in \widehat{\mathcal{W}}_{\mathbf{J}}^{*}$, an injective mapping $\phi_{\boldsymbol{s}} \colon \mathbb{N}_{\ell(\boldsymbol{s})} \to \Delta_{\mathbf{J}+}^{re}$ is defined by setting

$$\phi_{\boldsymbol{s}}(p) := [\boldsymbol{s}|_{\kappa(p)-1}](\alpha_{\boldsymbol{s}(\kappa(p))})$$

for each $p \in \mathbb{N}$, where the κ is a unique strictly increasing function $\kappa \colon \mathbb{N}_{\ell(\boldsymbol{s})} \to \mathbb{N}$ such that the image of κ equals to $\boldsymbol{s}^{-1}(S_{\mathbf{J}})$, i.e., $\mathrm{Im}(\kappa) = \boldsymbol{s}^{-1}(S_{\mathbf{J}})$. We denote by $\Phi_{\mathbf{J}}^{\ell(\boldsymbol{s})}(\boldsymbol{s})$ the image of the injective mapping $\phi_{\boldsymbol{s}}$. Note that if $\ell(\boldsymbol{s}) < \infty$ then $\Phi_{\mathbf{J}}^{\ell(\boldsymbol{s})}(\boldsymbol{s}) = \Phi_{\mathbf{J}}([\boldsymbol{s}])$.

For a pair $(\boldsymbol{s}, \boldsymbol{s}')$ of elements of $\widehat{\mathcal{W}}_{\mathbf{J}}^{\infty}$, we write $\boldsymbol{s} \sim \boldsymbol{s}'$ if for each $(p, q) \in \mathbb{N}^2$ there exists $(p_0, q_0) \in \mathbb{Z}_{\geq p} \times \mathbb{Z}_{\geq q}$ such that

$$\ell_{\mathbf{J}}([\boldsymbol{s}|_{p}]|^{-1}[\boldsymbol{s}'|_{p_0}]) = p_0 - p, \qquad \ell_{\mathbf{J}}([\boldsymbol{s}'|_{q}]^{-1}[\boldsymbol{s}|_{q_0}]) = q_0 - q.$$

Then we see that $\sim$ is an equivalence relation on $\widehat{\mathcal{W}}_{\mathbf{J}}^{\infty}$ (cf. [7]). We denote by $\widehat{W}_{\mathbf{J}}^{\infty}$ the quotient set of $\widehat{\mathcal{W}}_{\mathbf{J}}^{\infty}$ relative to the equivalence relation $\sim$, and by $[\boldsymbol{s}]$ the coset containing $\boldsymbol{s} \in \widehat{\mathcal{W}}_{\mathbf{J}}^{\infty}$. Let $W_{\mathbf{J}}^{\infty}$ be the image of $\mathcal{W}_{\mathbf{J}}^{\infty}$ by the canonical mapping $\widehat{\mathcal{W}}_{\mathbf{J}}^{\infty} \to \widehat{W}_{\mathbf{J}}^{\infty}$. Then we can easily show that $W_{\mathbf{J}}^{\infty} = \widehat{W}_{\mathbf{J}}^{\infty}$. Moreover, we see that $\boldsymbol{s} \sim \boldsymbol{s}'$ if and only if $\Phi_{\mathbf{J}}^{\infty}(\boldsymbol{s}) = \Phi_{\mathbf{J}}^{\infty}(\boldsymbol{s}')$ (cf. [7]). Hence we may denote by $\Phi_{\mathbf{J}}^{\infty}([\boldsymbol{s}])$ the set $\Phi_{\mathbf{J}}^{\infty}(\boldsymbol{s})$.

In the case where $\mathbf{J} = \mathring{\mathbf{I}}$, we will denote the symbols above more simply by removing $\mathbf{J}$ from them.

Theorem 3.1 ([7])**.** *For each non-empty subset* $\mathbf{J} \subset \mathring{\mathbf{I}}$*, let* $\Phi_{\mathbf{J}}^{*}$ *be the mapping from* $W_{\mathbf{J}}^{*}$ *to* $\mathfrak{B}_{\mathbf{J}}^{*}$ *defined by setting*

$$\Phi_{\mathbf{J}}^{*}([\boldsymbol{s}]) := \Phi_{\mathbf{J}}([\boldsymbol{s}]) \quad \textit{for } \boldsymbol{s} \in \mathcal{W}_{\mathbf{J}}, \qquad \Phi_{\mathbf{J}}^{*}([\boldsymbol{s}]) := \Phi_{\mathbf{J}}^{\infty}([\boldsymbol{s}]) \quad \textit{for } \boldsymbol{s} \in \mathcal{W}_{\mathbf{J}}^{\infty}.$$

Then $\Phi_{\mathbf{J}}^{*}$ *is a bijective mapping from* $W_{\mathbf{J}}^{*}$ *to* $\mathfrak{B}_{\mathbf{J}}^{*}$*, which maps* $W_{\mathbf{J}}^{\infty}$ *onto* $\mathfrak{B}_{\mathbf{J}}^{\infty}$*.*

Definition 3.2. Let $\preceq$ be a total order on a subset B of $\Delta_{\mathbf{J}+}$. We say that $\preceq$ is a *convex order* on B if it satisfies the following conditions:

CO(i) $(\beta, \gamma) \in B^2 \setminus (\Delta_{+}^{im})^2, \ \beta \prec \gamma, \ \beta + \gamma \in B \implies \beta \prec \beta + \gamma \prec \gamma\,;$

CO(ii) $\beta \in B, \ \gamma \in \Delta_{\mathbf{J}+} \setminus B, \ \beta + \gamma \in B \implies \beta \prec \beta + \gamma.$

Here we write $\beta \prec \gamma$ if $\beta \preceq \gamma$ and $\beta \neq \gamma$. We denote by $\preceq^{op}$ the total order on B defined by setting $\beta \preceq^{op} \gamma \Leftrightarrow \beta \succeq \gamma$ for each pair $(\beta, \gamma) \in B^2$, and call $\preceq^{op}$ the opposite of $\preceq$. We also say that $\preceq$ is an *opposite convex order* if the opposite $\preceq^{op}$ is a convex order.

For subsets C and D of B, we write $C \prec D$ if $c \prec d$ for all pair $(c, d) \in C \times D$.

For each non-empty subset $\mathbf{J} \subset \mathring{\mathbf{I}}$, we set

$$\mathcal{C}_n\mathbf{J} := \{\, \mathbf{K}_{\bullet} = (\mathbf{K}_0, \mathbf{K}_1, \ldots, \mathbf{K}_n) \,|\, \mathbf{J} = \mathbf{K}_0 \supsetneq \mathbf{K}_1 \supsetneq \cdots \supsetneq \mathbf{K}_n = \emptyset \,\}.$$

We note that if $n > \sharp\mathbf{J}$ then $\mathcal{C}_n\mathbf{J} = \emptyset$, and set

$$\mathcal{C}\mathbf{J} := \coprod_{n=1}^{\sharp\mathbf{J}} \mathcal{C}_n\mathbf{J}.$$

For each $n \in \mathbb{N}_{\sharp\mathbf{J}}$ and $\mathbf{K}_\bullet \in \mathcal{C}_n\mathbf{J}$, we set

$$W_{\mathbf{K}_\bullet} := W_{\mathbf{K}_1} \times \cdots \times W_{\mathbf{K}_n}, \quad \mathcal{W}^\infty_{\mathbf{K}_\bullet} := \mathcal{W}^\infty_{\mathbf{K}_0} \times \cdots \times \mathcal{W}^\infty_{\mathbf{K}_{n-1}}.$$

Denote an element $(y_1, \ldots, y_n) \in W_{\mathbf{K}_\bullet}$ by $y_\bullet$, and an element $(\boldsymbol{s}_0, \ldots, \boldsymbol{s}_{n-1}) \in \mathcal{W}^\infty_{\mathbf{K}_\bullet}$ by $\boldsymbol{s}_\bullet$. Note that $W_{\mathbf{K}_n} = \{1\}$ and $y_n = 1$ for each $\mathbf{K}_\bullet \in \mathcal{C}_n\mathbf{J}$ and $y_\bullet \in W_{\mathbf{K}_\bullet}$.

Theorem 3.3 ([8])**.** *Let $\mathbf{J}$ be an arbitrary non-empty subset of $\mathring{\mathbf{I}}$, and w an arbitrary element of $\mathring{W}_\mathbf{J}$.*

(1) *Let $\preceq_-$ be an arbitrary convex order on $\Delta_\mathbf{J}(w,-)$, $\preceq_0$ an arbitrary total order on Δ^{im}_+, and $\preceq_+$ an arbitrary opposite convex order on $\Delta_\mathbf{J}(w,+)$. We can define a convex order $\preceq$ on $\Delta_{\mathbf{J}+}$ by extending $\preceq_-, \preceq_0, \preceq_+$ to $\Delta_{\mathbf{J}+} = \Delta_\mathbf{J}(w,-) \amalg \Delta^{im}_+ \amalg \Delta_\mathbf{J}(w,+)$ in such a way that*

$$\Delta_\mathbf{J}(w,-) \prec \Delta^{im}_+ \prec \Delta_\mathbf{J}(w,+).$$

Moreover, we can obtain every convex orders on $\Delta_{\mathbf{J}+}$ by applying the procedure above.

(2) *For each $n \in \mathbb{N}_{\sharp\mathbf{J}}$ and $\mathbf{K}_\bullet \in \mathcal{C}_n\mathbf{J}$, there exists $(y_\bullet, \boldsymbol{s}_\bullet) \in W_{\mathbf{K}_\bullet} \times \mathcal{W}^\infty_{\mathbf{K}_\bullet}$ such that*

$$\Delta_\mathbf{J}(w,-) = \amalg_{i=1}^n w^{\mathbf{K}_{i-1}} y_{i-1} \Phi^\infty_{\mathbf{K}_{i-1}}([\boldsymbol{s}_{i-1}]), \tag{3.1}$$

$$C_i := \amalg_{j=1}^i w^{\mathbf{K}_{j-1}} y_{j-1} \Phi^\infty_{\mathbf{K}_{j-1}}([\boldsymbol{s}_{j-1}]) \in \mathfrak{B}^\infty_\mathbf{J} \quad \textit{for each } 1 \leqq i \leqq n, \tag{3.2}$$

where $y_0 := 1$. Then we can define a convex order $\preceq$ on $\Delta_\mathbf{J}(w,-)$ by applying the following procedure Steps 1,2.

Step 1. *For each $i = 1, \ldots, n$, define a total order $\preceq_i$ on the following set*

$$R_i := w^{\mathbf{K}_{i-1}} y_{i-1} \Phi^\infty_{\mathbf{K}_{i-1}}([\boldsymbol{s}_{i-1}])$$

by setting

$$w^{\mathbf{K}_{i-1}} y_{i-1} \phi_{\boldsymbol{s}_{i-1}}(p) \preceq_i w^{\mathbf{K}_{i-1}} y_{i-1} \phi_{\boldsymbol{s}_{i-1}}(q) \quad \textit{for each } p \leq q.$$

Step 2. *Define $\preceq$ by extending $\preceq_1, \ldots, \preceq_n$ to $\Delta_\mathbf{J}(w,-) = \amalg_{i=1}^n R_i$ in such a way that*

$$R_i \prec R_{i'} \quad \textit{for each } i < i'.$$

Moreover, we can obtain every convex orders on $\Delta_\mathbf{J}(w,-)$ by applying the procedure above.

Remark 3.4. (1) *Theorem 3.3 gives a concrete method of constructing all convex order on $\Delta_{\mathbf{J}+}$, since $\Delta_\mathbf{J}(w,+) = \Delta_\mathbf{J}(ww_\circ,-)$ with $w_\circ$ the longest element of $\mathring{W}_\mathbf{J}$.*

(2) *For each $n \in \mathbb{N}_{\sharp\mathbf{J}}$, we call the convex order on $\Delta_\mathbf{J}(w,-)$ described above that of n-row type.*

Definition 3.5. Let us use the notations as in Proposition 2.5 and Lemma 2.7(5). From now on, we often denote the translation t_{ε_j} $(j \in \mathring{\mathbf{I}})$ simply by ε_j if there is no fear of misunderstanding. Let $\mathbf{J}$ be an arbitrary non-empty subset of $\mathring{\mathbf{I}}$. For each $s \in \widehat{S}_{\mathbf{J}}$, we define an element $\widetilde{s} \in \widehat{W}$ by setting

$$\widetilde{s} := \begin{cases} (\varepsilon_j)^{\mathbf{J}_c} & \text{if } s = \rho_{\mathbf{J}_c j} \text{ with } c = 1, \dots, \mathrm{C}(\mathbf{J}) \text{ and } j \in \mathbf{J}_{c*}, \\ s_j & \text{if } s = s_j \text{ with } j \in \mathbf{J}, \\ (\varepsilon_{j_c})^{\mathbf{J}_c} s_{j_c^-} (\varepsilon_{j_c^-})^{\mathbf{J}_c} & \text{if } s = s_{\delta - \theta_{\mathbf{J}_c}} \text{ with } c = 1, \dots, \mathrm{C}(\mathbf{J}), \end{cases} \tag{3.3}$$

where we fix an element of $j_c \in \mathbf{J}_{c*}$ for each $c = 1, \dots, \mathrm{C}(\mathbf{J})$. For each $\rho \in \Omega_{\mathbf{J}}$, we define an element $\widetilde{\rho} \in \widehat{W}$ by setting

$$\widetilde{\rho} := \textstyle\prod_{c=1}^{\mathrm{C}(\mathbf{J})} \widetilde{\rho}_c, \tag{3.4}$$

where $\rho = \prod_{c=1}^{\mathrm{C}(\mathbf{J})} \rho_c$ with $\rho_c \in \Omega_{\mathbf{J}_c}$, and if $\rho_c = 1$ then we set $\widetilde{\rho}_c := 1$.

For each $s \in \widehat{S}_{\mathbf{J}}$, we fix a finite reduced word $\boldsymbol{r}_s = (\boldsymbol{r}_s(p))_{p \in \mathbb{N}_{N_s}} \in \widehat{\mathcal{W}}$ such that $[\boldsymbol{r}_s] = \widetilde{s}$, where $\boldsymbol{r}_s(p) \in \widehat{S}$ for all $p \in \mathbb{N}_{N_s}$. For each $\boldsymbol{s} = (\boldsymbol{s}(p))_{p \in \mathbb{N}_n} \in \widehat{\mathcal{W}}^*_{\mathbf{J}}$ with $n \in \mathbb{N}_*$, we define a sequence $\widetilde{\boldsymbol{s}} = (\widetilde{\boldsymbol{s}}(p))_{p \in \mathbb{N}_{\widetilde{n}}} \in \widehat{S}_{\mathbf{J}}^{\mathbb{N}_{\widetilde{n}}}$ with $\widetilde{n} \in \mathbb{N}_*$ by setting

$$\widetilde{\boldsymbol{s}} := \begin{cases} \boldsymbol{r}_{s_1} \boldsymbol{r}_{s_2} \cdots \boldsymbol{r}_{s_n} & \text{if } n < \infty, \\ \lim_{p \to \infty} \boldsymbol{r}_{s_1} \boldsymbol{r}_{s_2} \cdots \boldsymbol{r}_{s_p} & \text{if } n = \infty, \end{cases} \tag{3.5}$$

where $s_p := \boldsymbol{s}(p)$ for each $p \in \mathbb{N}_n$. Here,

$$\widetilde{n} := \begin{cases} N_{s_1} + N_{s_2} + \dots + N_{s_n} & \text{if } n < \infty, \\ \infty & \text{if } n = \infty. \end{cases}$$

Note that for each $p \in \mathbb{N}_n$,

$$[\widetilde{\boldsymbol{s}|_p}] = \widetilde{\boldsymbol{s}(1)} \widetilde{\boldsymbol{s}(2)} \cdots \widetilde{\boldsymbol{s}(p)}. \tag{3.6}$$

Lemma 3.6. (1) *The sequence* $\widetilde{\boldsymbol{s}} = (\widetilde{\boldsymbol{s}}(p))_{p \in \mathbb{N}_{\widetilde{n}}}$ *defined in Definition 3.5 is an element of* $\widehat{\mathcal{W}}^*$ *such that* $\phi_{\widetilde{\boldsymbol{s}}} \circ f = \phi_{\boldsymbol{s}}$ *for some unique strictly increasing function* $f \colon \mathbb{N}_{\ell(\boldsymbol{s})} \to \mathbb{N}$. *In particular,* $\widetilde{\boldsymbol{s}} \in \widehat{\mathcal{W}}^\infty$ *if and only if* $\boldsymbol{s} \in \widehat{\mathcal{W}}^\infty_{\mathbf{J}}$. *Moreover, the following equalities hold*:

$$[\widetilde{\boldsymbol{s}|_p}]|_{\mathfrak{h}'^*_{\mathbf{J}}} = [\boldsymbol{s}|_p], \tag{3.7}$$

$$[\widetilde{\boldsymbol{s}|_p}]\, \Delta^{\mathbf{J}}(1, -) \subset \Delta^{\mathbf{J}}(1, -), \tag{3.8}$$

$$\Phi([\widetilde{\boldsymbol{s}|_p}]) \cap \Delta_{\mathbf{J}+} = \Phi_{\mathbf{J}}([\boldsymbol{s}|_p]), \tag{3.9}$$

$$\Phi([\widetilde{\boldsymbol{s}|_p}]) \setminus \Phi_{\mathbf{J}}([\boldsymbol{s}|_p]) \subset \Delta^{\mathbf{J}}(1, -), \tag{3.10}$$

$$\ell([\widetilde{\boldsymbol{s}|_p}]) = \textstyle\sum_{k=1}^{p} \ell(\widetilde{\boldsymbol{s}(k)}) \tag{3.11}$$

for all $p \in \mathbb{N}_n$. *In particular,*

$$\Phi^*([\widetilde{\boldsymbol{s}}]) \cap \Delta_{\mathbf{J}+} = \Phi^*_{\mathbf{J}}([\boldsymbol{s}]), \tag{3.12}$$

$$\Phi^*([\widetilde{\boldsymbol{s}}]) \setminus \Phi^*_{\mathbf{J}}([\boldsymbol{s}]) \subset \Delta^{\mathbf{J}}(1, -). \tag{3.13}$$

(2) *If* $\boldsymbol{s} \in \widehat{\mathcal{W}}^\infty_{\mathbf{J}}$, *then* $\widetilde{\boldsymbol{s}} \in \widehat{\mathcal{W}}^\infty$ *with the following equality*:

$$\Phi^\infty([\widetilde{\boldsymbol{s}}]) = \Phi^\infty_{\mathbf{J}}([\boldsymbol{s}]) \amalg \Delta^{\mathbf{J}}(1, -). \tag{3.14}$$

Proof. Let us prove the part (1). The (3.7) follows from (2.14). By (2.12), we see that $\widetilde{\boldsymbol{s}(k)} \Delta^{\mathbf{J}}(1, -) \subset \Delta^{\mathbf{J}}(1, -)$ for all $k \in \mathbb{N}_p$. Thus, by (3.6) we get (3.8). By (2.10), (2.15), and (2.16), we have

$$\Phi(\widetilde{\boldsymbol{s}(k)}) \cap \Delta_{\mathbf{J}+} = \Phi_{\mathbf{J}}(\boldsymbol{s}(k)), \qquad \Phi(\widetilde{\boldsymbol{s}(k)}) \setminus \Phi_{\mathbf{J}}(\boldsymbol{s}(k)) \subset \Delta^{\mathbf{J}}(1, -). \tag{3.15}$$

By (3.6)(3.7)(3.8) and (3.15), we have

$$\Phi([\widetilde{\boldsymbol{s}_{|p}}]) = \textstyle\coprod_{k=1}^{p}[\widetilde{\boldsymbol{s}_{|k-1}}]\Phi(\widetilde{\boldsymbol{s}_{(k)}}), \tag{3.16}$$

where $[\widetilde{\boldsymbol{s}_{|0}}] = 1$. Therefore we see that (3.11) holds and the sequence $\widetilde{\boldsymbol{s}}$ is an element of $\widehat{\mathcal{W}}^*$ satisfying (3.9) and (3.10). It is easy to see that $\Phi^*([\widetilde{\boldsymbol{s}}]) = \cup_{p\in\mathbb{N}_n}\Phi([\widetilde{\boldsymbol{s}_{|p}}])$. Hence (3.9) and (3.10) imply (3.12) and (3.13). By (3.7), (3.16), and the left equation in (3.15), we see that there exists a unique strictly increasing function $f\colon \mathbb{N}_{\ell(\boldsymbol{s})} \to \mathbb{N}$ such that $\phi_{\widetilde{\boldsymbol{s}}} \circ f = \phi_{\boldsymbol{s}}$.

Let us prove the part (2). By the part (1), the $\widetilde{\boldsymbol{s}}$ is an element of $\widehat{\mathcal{W}}^\infty$. In the case where $\mathbf{J} = \mathring{\mathbf{I}}$, we see that $\Delta^{\mathbf{J}}(1,-) = \emptyset$ and $\widetilde{\boldsymbol{s}} = \boldsymbol{s}$, and hence the equality (3.14) is valid. Suppose that $\mathbf{J}$ is a proper subset of $\mathring{\mathbf{I}}$. By (3.12) and (3.13), and Theorem 3.1, we see that the set $\Phi^\infty([\widetilde{\boldsymbol{s}}])$ is an infinite real biconvex set such that $\Phi^\infty([\widetilde{\boldsymbol{s}}]) \cap \Delta_{\mathbf{J}+} = \Phi^\infty_{\mathbf{J}}([\boldsymbol{s}])$ and the set $\Phi^\infty([\widetilde{\boldsymbol{s}}]) \setminus \Phi^\infty_{\mathbf{J}}([\boldsymbol{s}])$ is an infinite subset of $\Delta^{\mathbf{J}}(1,-)$, which implies the equality (3.14) by Theorem 2.4. □

For each $B \in \mathfrak{B}^*_{\mathbf{J}}$, we set

$$\widehat{W}_{\mathbf{J}}(B) := \{\, y \in \widehat{W}_{\mathbf{J}} \mid \Phi_{\mathbf{J}}(y) \subset B \,\}, \quad W_{\mathbf{J}}(B) := \widehat{W}_{\mathbf{J}}(B) \cap W_{\mathbf{J}}. \tag{3.17}$$

Lemma 3.7. (1) *Let B be a real biconvex set in $\Delta_{\mathbf{J}+}$. Then, for each pair $(y_1, y_2) \in \widehat{W}_{\mathbf{J}}(B)^2$, there exists an element $y_3 \in \widehat{W}_{\mathbf{J}}(B)$ such that $\Phi_{\mathbf{J}}(y_1) \cup \Phi_{\mathbf{J}}(y_2) \subset \Phi_{\mathbf{J}}(y_3)$.*

(2) *Suppose that a subset $Y \subset \widehat{W}_{\mathbf{J}}$ satisfies the condition: for each pair $(y_1, y_2) \in Y^2$, there exists an element $y_3 \in Y$ such that $\Phi_{\mathbf{J}}(y_1) \cup \Phi_{\mathbf{J}}(y_2) \subset \Phi_{\mathbf{J}}(y_3)$. Then the set $\Phi_{\mathbf{J}}(Y)$ below is a real biconvex set in $\Delta_{\mathbf{J}+}$:*

$$\Phi_{\mathbf{J}}(Y) := \bigcup_{y\in Y} \Phi_{\mathbf{J}}(y). \tag{3.18}$$

Proof. We prove the part (1). By Theorem 3.1, we have $B = \Phi^*_{\mathbf{J}}([\boldsymbol{s}])$ for some $\boldsymbol{s} \in \mathcal{W}^*_{\mathbf{J}}$, hence $B = \cup_{p=1}^{\ell(\boldsymbol{s})}\Phi_{\mathbf{J}}([\boldsymbol{s}_{|p}])$. Since $\Phi_{\mathbf{J}}(y_1) \cup \Phi_{\mathbf{J}}(y_2)$ is a finite set and $\Phi_{\mathbf{J}}([\boldsymbol{s}_{|p}]) \subsetneq \Phi_{\mathbf{J}}([\boldsymbol{s}_{|p'}])$ for $p < p'$, we see that $\Phi_{\mathbf{J}}(y_1) \cup \Phi_{\mathbf{J}}(y_2) \subset \Phi_{\mathbf{J}}([\boldsymbol{s}_{|p_0}])$ for some $p_0 \in \mathbb{N}_{\ell(\boldsymbol{s})}$.

We prove the part (2). Suppose that $\beta, \gamma \in \Phi_{\mathbf{J}}(Y)$ satisfy $\beta + \gamma \in \Delta_{\mathbf{J}+}$. By the assumption of Y, we may assume that $\beta, \gamma \in \Phi_{\mathbf{J}}(y)$ for some $y \in Y$. Then $\beta+\gamma \in \Phi_{\mathbf{J}}(y)$, hence $\beta+\gamma \in \Phi_{\mathbf{J}}(Y)$. It is clear that $\Delta_{\mathbf{J}} \setminus \Phi_{\mathbf{J}}(Y) = \cap_{y\in Y}\{\Delta_{\mathbf{J}} \setminus \Phi_{\mathbf{J}}(y)\}$. Suppose that $\beta, \gamma \in \Delta_{\mathbf{J}} \setminus \Phi_{\mathbf{J}}(Y)$ satisfy $\beta + \gamma \in \Delta_{\mathbf{J}+}$. Then $\beta, \gamma \in \Delta_{\mathbf{J}} \setminus \Phi_{\mathbf{J}}(y)$ for all $y \in Y$. It follows that $\beta+\gamma \in \Delta_{\mathbf{J}} \setminus \Phi_{\mathbf{J}}(y)$ for all $y \in Y$, hence $\beta+\gamma \in \Delta_{\mathbf{J}} \setminus \Phi_{\mathbf{J}}(Y)$. □

Proposition 3.8. *We use the notations introduced in Definition 2.3 and Definition 3.5. Let (J, u, y) be an arbitrary element of $\boldsymbol{\mathcal{P}}$. Suppose that $\varepsilon \in \mathring{P}^\vee$ satisfies $(\varepsilon \,|\, \alpha_i) > 0$ for all $i \in \mathring{\mathbf{I}} \setminus \mathbf{J}$ and $(\varepsilon \,|\, \alpha_j) = 0$ for all $j \in \mathbf{J}$ and that $\boldsymbol{s} \in \mathcal{W}_{\mathbf{J}}$ satisfies $[\boldsymbol{s}] = y$. Then we have*

$$\nabla(\mathrm{J}, u, y) = \bigcup_{n\geq 0} \Phi(u[\widetilde{\boldsymbol{s}}]t^n_\varepsilon). \tag{3.19}$$

Proof. Set $B = \cup_{n\geq 0}\Phi([\widetilde{\boldsymbol{s}}]t^n_\varepsilon)$. By the assumption of ε, we see that $\cup_{n\geq 0}\Phi(t^n_\varepsilon) = \Delta^{\mathbf{J}}(1,-)$. Hence, by (3.8) and Lemma 2.3(2) in [7], we see that $\Phi([\widetilde{\boldsymbol{s}}]t^n_\varepsilon) = \Phi([\widetilde{\boldsymbol{s}}]) \amalg [\widetilde{\boldsymbol{s}}]\Phi(t^n_\varepsilon)$ for all $n \geq 0$. Thus, by Lemma 3.7(2), we see that B is an infinite real biconvex set such that

$$B = \Phi([\widetilde{\boldsymbol{s}}]) \amalg [\widetilde{\boldsymbol{s}}]\Delta^{\mathbf{J}}(1,-).$$

Hence, by (3.8)(3.9)(3.10) we have $B \cap \Delta_{\mathbf{J}+} = \Phi_{\mathbf{J}}(y)$ and $B \setminus \Phi_{\mathbf{J}}(y) \subset \Delta^{\mathbf{J}}(1,-)$. Since $\Delta^{\mathbf{J}}(1,-) \setminus [\widetilde{\boldsymbol{s}}]\Delta^{\mathbf{J}}(1,-)$ is a finite set, we see that $\Delta^{\mathbf{J}}(1,-) \setminus B$ is a finite set. By Theorem 2.4, we get $B = \Phi_{\mathbf{J}}(y) \amalg \Delta^{\mathbf{J}}(1,-)$. Since $u \in \mathring{W}^{\mathbf{J}}$ we see that $\Phi(u[\widetilde{\boldsymbol{s}}]t_\varepsilon^n) = \Phi(u) \amalg u\Phi([\widetilde{\boldsymbol{s}}]t_\varepsilon^n)$ for all $n \geq 0$, which implies that

$$\begin{aligned}\cup_{n\geq 0}\Phi(u[\widetilde{\boldsymbol{s}}]t_\varepsilon^n) &= \Phi(u) \amalg u \cup_{n\geq 0} \Phi([\widetilde{\boldsymbol{s}}]t_\varepsilon^n)\\ &= \Phi(u) \amalg uB = u\Phi_{\mathbf{J}}(y) \amalg \Delta^{\mathbf{J}}(u,-) = \nabla(\mathbf{J},u,y).\end{aligned}$$

□

Lemma 3.9. *Let* $\mathbf{J}$ *and* $\mathbf{K}$ *be connected subsets of* $\mathring{\mathbf{I}}$ *such that* $\mathbf{K} \subset \mathbf{J}$, *and* k *an element of* $\mathbf{K}_*$. *Suppose that* $[\boldsymbol{s}] = t_{\varepsilon_k}$ *with* $\boldsymbol{s} \in \widehat{\mathcal{W}}_{\mathbf{J}}$, *and denote the elements* $[\widetilde{\boldsymbol{s}}]$ *and* t_{ε_k} *of* $\widehat{W}$ *uniquely by* $[\widetilde{\boldsymbol{s}}] = [\widetilde{\boldsymbol{s}}]^{\mathbf{K}}[\widetilde{\boldsymbol{s}}]_{\mathbf{K}}$ *and* $t_{\varepsilon_k} = (t_{\varepsilon_k})^{\mathbf{K}}(t_{\varepsilon_k})_{\mathbf{K}}$ *with* $[\widetilde{\boldsymbol{s}}]^{\mathbf{K}} \in \widehat{W}^{\mathbf{K}}$, $(t_{\varepsilon_k})^{\mathbf{K}} \in \widehat{W}^{\mathbf{K}}$, $[\widetilde{\boldsymbol{s}}]_{\mathbf{K}} \in \mathring{W}_{\mathbf{K}}$, *and* $(t_{\varepsilon_k})_{\mathbf{K}} \in \mathring{W}_{\mathbf{K}}$. *Then the following equalities hold*:

$$\text{(i)}\ \ [\widetilde{\boldsymbol{s}}]^{\mathbf{K}}|_{\mathfrak{h}'^*_{\mathbf{K}}} = (t_{\varepsilon_k})^{\mathbf{K}}|_{\mathfrak{h}'^*_{\mathbf{K}}}, \qquad \text{(ii)}\ \ [\widetilde{\boldsymbol{s}}]_{\mathbf{K}} = (t_{\varepsilon_k})_{\mathbf{K}}. \tag{3.20}$$

Proof. By (3.7), we have $[\widetilde{\boldsymbol{s}}]|_{\mathfrak{h}'^*_{\mathbf{J}}} = t_{\varepsilon_k}|_{\mathfrak{h}'^*_{\mathbf{J}}}$, hence $[\widetilde{\boldsymbol{s}}]|_{\mathfrak{h}'^*_{\mathbf{K}}} = t_{\varepsilon_k}|_{\mathfrak{h}'^*_{\mathbf{K}}}$ since $\mathfrak{h}'^*_{\mathbf{K}} \subset \mathfrak{h}'^*_{\mathbf{J}}$. On the other hand, we see that $[\widetilde{\boldsymbol{s}}]|_{\mathfrak{h}'^*_{\mathbf{K}}} = [\widetilde{\boldsymbol{s}}]^{\mathbf{K}}|_{\mathfrak{h}'^*_{\mathbf{K}}}[\widetilde{\boldsymbol{s}}]_{\mathbf{K}}$ and $t_{\varepsilon_k}|_{\mathfrak{h}'^*_{\mathbf{K}}} = (t_{\varepsilon_k})^{\mathbf{K}}|_{\mathfrak{h}'^*_{\mathbf{K}}}(t_{\varepsilon_k})_{\mathbf{K}}$. Therefore (3.20) follows from Lemma 2.6. □

4. Notations and preliminary results for U_q

In this section, we prepare notations for the quantized enveloping algebra $U_q(\mathfrak{g})$ associated with the symmetrizable Kac-Moody Lie algebras $\mathfrak{g}$.

For each $n \in \mathbb{N}$, we define $[n]_t, [n]_t!, (n)_t, (n)_t! \in \mathbb{Z}[t,t^{-1}]$ by setting

$$[n]_t := \tfrac{t^n - t^{-n}}{t - t^{-1}}, \quad [n]_t! := \textstyle\prod_{k=1}^n [k]_t, \quad (n)_t := \tfrac{t^{2n}-1}{t^2-1}, \quad (n)_t! := \textstyle\prod_{k=1}^n (k)_t,$$

and set $[0]_t = (0)_t = [0]_t! = (0)_t! := 1$.

We assume that q is an indeterminate over $\mathbb{Q}$. Let $\mathbb{Q}(q)$ be the field of rational functions of q with coefficients in $\mathbb{Q}$. Let $U = U_q(\mathfrak{g})$ be the quantized enveloping algebra over $\mathbb{Q}(q)$ of the symmetrizable Kac-Moody Lie algebras $\mathfrak{g}$, that is, the associative $\mathbb{Q}(q)$-algebra U with the unit 1 defined by the generators $\{E_i, F_i \,|\, i \in \mathbf{I}\} \amalg \{K_\lambda \,|\, \lambda \in P\}$ and the following fundamental relations:

$$K_\lambda K_\mu = K_{\lambda+\mu}, \quad K_0 = 1, \tag{4.1}$$

$$K_\lambda E_i K_\lambda^{-1} = q^{(\alpha_i|\lambda)} E_i, \quad K_\lambda F_i K_\lambda^{-1} = q^{-(\alpha_i|\lambda)} F_i, \tag{4.2}$$

$$[E_i, F_j] = \delta_{ij}(K_i - K_i^{-1})/(q_i - q_i^{-1}), \tag{4.3}$$

$$\sum_{k=0}^{1-\mathrm{A}_{ij}} (-1)^k E_i^{(1-\mathrm{A}_{ij}-k)} E_j E_i^{(k)} = \sum_{k=0}^{1-\mathrm{A}_{ij}} (-1)^k F_i^{(1-\mathrm{A}_{ij}-k)} F_j F_i^{(k)} = 0 \quad (i \neq j), \tag{4.4}$$

where $q_i := q^{d_i}$, $K_i := K_{\alpha_i}$, $E_i^{(k)} = E_i^k/[k]_{q_i}!$, and $F_i^{(k)} = F_i^k/[k]_{q_i}!$. The relations (4.4) is called the quantum Serre relations. Let U' be the $\mathbb{Q}(q)$-subalgebra of U generated by $\{E_i, F_i, K_i^{\pm 1} \,|\, i \in \mathbf{I}\}$, U^+ the $\mathbb{Q}(q)$-subalgebra of U generated by $\{E_i \,|\, i \in \mathbf{I}\}$, U^- the $\mathbb{Q}(q)$-subalgebra of U generated by $\{F_i \,|\, i \in \mathbf{I}\}$, and U^0 the $\mathbb{Q}(q)$-subalgebra of U generated by $\{K_\lambda \,|\, \lambda \in P\}$. The multiplication mapping defines the following isomorphism of $\mathbb{Q}(q)$-vector spaces:

$$U^+ \otimes U^0 \otimes U^- \xrightarrow{\sim} U, \quad x \otimes y \otimes z \mapsto xyz. \tag{4.5}$$

We define $U^{\geq 0}$ and $U^{\leq 0}$ to be the images of $U^+ \otimes U^0$ and $U^0 \otimes U^-$ by the mapping (4.5), respectively. It is clear that both $U^{\geq 0}$ and $U^{\leq 0}$ are $\mathbb{Q}(q)$-subalgebras of U. Let $\Omega\colon U \to U$ be a $\mathbb{Q}$-algebra anti-automorphism such that

$$\Omega(E_i) = F_i, \quad \Omega(F_i) = E_i, \quad \Omega(K_\lambda) = K_\lambda^{-1}, \quad \Omega(q) = q^{-1}. \tag{4.6}$$

Let $\Psi\colon U \to U$ be a $\mathbb{Q}(q)$-algebra anti-automorphism such that

$$\Psi(E_i) = E_i, \quad \Psi(F_i) = F_i, \quad \Psi(K_\lambda) = K_\lambda^{-1}. \tag{4.7}$$

Note that both Ω and Φ stabilize U' and that the following equalities hold:

$$\Omega^2 = \Psi^2 = id_U, \quad [\Omega\,,\Phi] = 0. \tag{4.8}$$

For each $\mu \in Q$, the weight space U_μ of U with μ weight is defined by setting

$$U_\mu := \{\, u \in U \mid K_\lambda u K_\lambda^{-1} = q^{(\mu|\lambda)} u \ \ (\forall \lambda \in P)\,\}. \tag{4.9}$$

Then $U_\lambda U_\mu \subset U_{\lambda+\mu}$ for $\lambda, \mu \in Q$ and the following weight space decomposition of U holds:

$$U = \oplus_{\mu \in Q} U_\mu. \tag{4.10}$$

We call a non-zero element u of U_μ a weight vector with weight μ and set $\mathrm{wt}(u) := \mu$. In addition, for each $\mu \in Q_+$, we set

$$U_\mu^+ := U_\mu \cap U^+, \quad U_{-\mu}^- := U_{-\mu} \cap U^-, \quad U_\mu^{\geq 0} := U_\mu^+ U^0, \quad U_{-\mu}^{\leq 0} := U_{-\mu}^- U^0. \tag{4.11}$$

Then the following weight space decompositions hold:

$$U^+ = \oplus_{\mu \in Q_+} U_\mu^+,\ \ U^- = \oplus_{\mu \in Q_+} U_{-\mu}^-,\ \ U^{\geq 0} = \oplus_{\mu \in Q_+} U_\mu^{\geq 0},\ \ U^{\leq 0} = \oplus_{\mu \in Q_+} U_{-\mu}^{\leq 0}. \tag{4.12}$$

For each $\mu, \nu \in Q$, $u \in U_\mu$, $v \in U_\nu$, we set

$$[u, v]_q := uv - q^{(\mu|\nu)} vu, \tag{4.13}$$

and define a $\mathbb{Q}(q)$-bilinear mapping $[\ \ ,\ \ ]_q\colon U \times U \to U$ by setting

$$(x, y) \mapsto [x, y]_q := \textstyle\sum_{\mu,\nu \in Q} [x_\mu, y_\nu]_q, \tag{4.14}$$

where $x = \sum_{\mu \in Q} x_\mu$ $(x_\mu \in U_\mu)$, $y = \sum_{\nu \in Q} y_\mu$ $(x_\mu \in U_\nu)$. The mapping $[\ \ ,\ \ ]_q$ is called the q-commutator or the q-bracket. For each $x \in U$, we define a $\mathbb{Q}(q)$-linear mapping $ad_q x\colon U \to U$ by setting

$$(ad_q x).y := [x, y]_q. \tag{4.15}$$

For each $\alpha, \beta \in \Delta\,(\alpha \neq \beta)$, $x \in U_\alpha$, $y \in U_\beta$, and $n \in \mathbb{Z}_+$, we see that

$$\frac{1}{[n]_{q_\alpha}!}(ad_q x)^n.y = \sum_{k=0}^{n} (-1)^k q_\alpha^{k(n-1+\mathrm{A}_{\alpha\beta})} x^{(n-k)} y x^{(k)}, \tag{4.16}$$

where $\mathrm{A}_{\alpha\beta} := \frac{2(\alpha|\beta)}{(\alpha|\alpha)} \in \mathbb{Z}$, $q_\alpha := q^{(\alpha|\alpha)/2}$, and $x^{(k)} := x^k/[k]_{q_\alpha}!$. Here we set

$$(ad_q x)^{(n)}.y := \frac{1}{[n]_{q_\alpha}!}(ad_q x)^n.y. \tag{4.17}$$

Note that the quantum Serre relations (4.4) can be written as

$$(ad_q E_i)^{(1-\mathrm{A}_{ij})}.E_j = (ad_q F_i)^{(1-\mathrm{A}_{ij})}.F_j = 0 \quad (i \neq j).$$

The braid group $\mathcal{B}_W = \langle T_i \,|\, i \in \mathbf{I} \rangle$ associated with the Weyl group W acts on U as a group of $\mathbb{Q}(q)$-algebra automorphisms of U via

$$T_i(E_i) = -F_iK_i, \quad T_i(E_j) = (ad_q E_i)^{(-\mathrm{A}_{ij})}.E_j \quad (i \neq j), \tag{4.18}$$

$$T_i(F_i) = -K_i^{-1}E_i, \quad T_i(F_j) = \Omega(T_i(E_j)) \quad (i \neq j), \tag{4.19}$$

$$T_i(K_\lambda) = K_{s_i(\lambda)} = K_\lambda K_i^{-\langle \alpha_i^\vee, \lambda\rangle}, \tag{4.20}$$

where $i, j \in \mathbf{I}$, $\lambda \in P$ (cf. [14]). Note that the following equalities hold:

$$\Omega T_i = T_i \Omega, \qquad \Psi T_i \Psi = T_i^{-1} \tag{4.21}$$

as automorphisms of U and that the subalgebra U' is stable under the action of $\mathcal{B}_W$ on U and the action is faithful. For each $x \in W$, we set

$$T_x := T_{i_1}T_{i_2}\cdots T_{i_n}, \tag{4.22}$$

where $x = s_{i_1}s_{i_2}\cdots s_{i_n}$ with $n = \ell(x)$ and $i_1, i_2, \ldots, i_n \in \mathbf{I}$ is a reduced expression of x. The T_x does not depend on reduced expressions of x.

Lemma 4.1 ([14])**.** *For each $i \in \mathbf{I}$ and integrable U-module M, define the $\mathbb{Z}$-gradation $M = \oplus_{n\in\mathbb{Z}} M_i^n$ by setting $M_i^n := \{m \in M \,|\, K_i.m = q_i^n m\}$ for each $n \in \mathbb{Z}$, and define $T_{iM}\colon M \to M$ to be the $\mathbb{Q}(q)$-linear mapping by setting*

$$T_{iM}(m) := \sum_{a,b,c\geq 0; -a+b-c=n} (-1)^b q_i^{b-ac} E_i^{(a)} F_i^{(b)} E_i^{(c)}.m \tag{4.23}$$

for each $n \in \mathbb{Z}$ and each $m \in M_i^n$. Then

$$T_i(u).T_{iM}(m) = T_{iM}(u.m) \tag{4.24}$$

for all $u \in U$ and all $m \in M$.

Each subgroup $\Omega \subset \mathrm{Aut}(\Delta, \Pi)$ acts faithfully on U' as a group of $\mathbb{Q}(q)$-algebra automorphisms of U' via

$$T_\rho(E_i) = E_{\rho(i)}, \quad T_\rho(F_i) = F_{\rho(i)}, \quad T_\rho(K_i) = K_{\rho(i)}, \tag{4.25}$$

where $\rho \in \Omega$ and $i \in \mathbf{I}$. The following equalities hold:

$$T_\rho T_i = T_{\rho(i)} T_\rho \tag{4.26}$$

as automorphisms of U. Set $\widehat{W} := W \rtimes \Omega$. Then the braid group $\mathcal{B}_{\widehat{W}} = \langle T_x \,|\, x \in \widehat{W}\rangle$ acts faithfully on U' as a group of $\mathbb{Q}(q)$-algebra automorphisms of U' via

$$T_x = T_{|x|} T_{\rho_x}, \tag{4.27}$$

where $x = |x|\rho_x$, $|x| \in W$, and $\rho_x \in \Omega$.

Let $\mathcal{A}_1$ be the localisation of the polynomial ring $\mathbb{Q}[q]$ at the maximal ideal $(q-1)$, that is, the $\mathbb{Q}$-subalgebra of $\mathbb{Q}(q)$ consisting of elements of $\mathbb{Q}(q)$ which have no pole at $q = 1$. For each $\mathcal{A}_1$-module M, we can define a vector space ${}_1M$ over $\mathbb{Q}$ by setting ${}_1M := \mathbb{Q} \otimes_{\mathcal{A}_1} M$, where $\mathbb{Q}$ is regarded as an $\mathcal{A}_1$-algebra via $q \mapsto 1$, and call the canonical mapping $M \to {}_1M$ *the specialization at* $q = 1$. We note that ${}_1M \simeq M/\{(q-1)M\}$, and denote by $\overline{m}$ the image of $m \in M$ by the specialization at $q = 1$.

Let ${}_{\mathcal{A}_1}U'$ be the $\mathcal{A}_1$-subalgebra of U' generated by $\{E_i, F_i, K_i^{\pm 1} \,|\, i \in \mathbf{I}\}$, and ${}_{\mathcal{A}_1}U^+$ the $\mathcal{A}_1$-subalgebra of U' generated by $\{E_i \,|\, i \in \mathbf{I}\}$. Note that ${}_{\mathcal{A}_1}U'$ is stable under the action of $\mathcal{B}_W$ on U. Set ${}_{\mathcal{A}_1}U^+_\mu := {}_{\mathcal{A}_1}U^+ \cap U^+_\mu$ for each $\mu \in Q_+$. Then ${}_{\mathcal{A}_1}U^+ = \oplus_{\mu\in Q_+} {}_{\mathcal{A}_1}U^+_\mu$. We denote simply by ${}_1U^+$ and ${}_1U^+_\mu$ the image of ${}_{\mathcal{A}_1}U^+$ and ${}_{\mathcal{A}_1}U^+_\mu$ by the specialization at $q = 1$, respectively. Since ${}_{\mathcal{A}_1}U^+_\mu$ is a finitely generated $\mathcal{A}_1$-module with torsion-free and $\mathcal{A}_1$ is a principal ideal domain, we see that ${}_{\mathcal{A}_1}U^+_\mu$ is a free $\mathcal{A}_1$-module with finite rank.

We define two sets $\tilde{\Delta}^{im}_+$ and $\tilde{\Delta}_+$ by setting

$$\tilde{\Delta}^{im}_+ := \{\, (\eta, i) \mid \eta \in \Delta^{im}_+,\ i = 1, \ldots, \mathrm{mult}(\eta) \,\}, \quad \tilde{\Delta}_+ := \Delta^{re}_+ \amalg \tilde{\Delta}^{im}_+, \tag{4.28}$$

where $\mathrm{mult}(\eta) := \dim_{\mathbb{Q}} \mathfrak{g}_\eta$, and define $\kappa\colon Q_+ \to \mathbb{N}$ by setting for each $\mu \in Q_+$,

$$\kappa(\mu) := \sharp\{\, \mathbf{c}\colon \tilde{\Delta}_+ \to \mathbb{Z}_+ \mid \textstyle\sum_{\alpha \in \Delta^{re}_+} \mathbf{c}(\alpha)\alpha + \sum_{\eta \in \Delta^{im}_+} \sum_{i=1}^{\mathrm{mult}(\eta)} \mathbf{c}((\eta, i))\eta = \mu \,\}. \tag{4.29}$$

Proposition 4.2 ([5],[14])**.** *The $\mathbb{Q}$-algebra ${}_1U^+$ is characterized as the associative $\mathbb{Q}$-algebra with the unit $\overline{1}$ defined by the generators $\{\overline{E}_i \,|\, i \in \mathbf{I}\}$ and the following fundamental relations*:

$$\sum_{k=0}^{1-\mathrm{A}_{ij}} (-1)^k \overline{E}_i^{(1-\mathrm{A}_{ij}-k)} \overline{E}_j \overline{E}_i^{(k)} = 0 \quad (i \neq j), \tag{4.30}$$

where $\overline{E}_i^{(k)} = \overline{E}_i^k / k!$. Moreover, for each $\mu \in Q_+$, the following equality holds:

$$\dim_{\mathbb{Q}} {}_1U^+_\mu = \dim_{\mathbb{Q}(q)} U^+_\mu = \mathrm{rank}_{\mathcal{A}_1}({}_{\mathcal{A}_1}U^+_\mu) = \kappa(\mu). \tag{4.31}$$

Lemma 4.3. *Let V be a vector space over $\mathbb{Q}(q)$, W a submodule of V over $\mathcal{A}_1$, and $X = \{\, x_\lambda \,|\, \lambda \in \Lambda \,\}$ is a subset of W with Λ an index set. Suppose that the elements of $\{\, \overline{x_\lambda} \,|\, \lambda \in \Lambda \,\}$ are linearly independent over $\mathbb{Q}$. Then the elements of X are linearly independent over $\mathbb{Q}(q)$. Here, $\overline{x_\lambda}$ is the image of x_λ by the specialization at $q = 1$. Moreover, if, in addition, the subset X is a basis of V, then X is a basis of W over $\mathcal{A}_1$.*

Proof. Suppose that $\sum_{\lambda \in \mathrm{L}} k_\lambda x_\lambda = 0$ for some finite subset $\mathrm{L} \subset \Lambda$ with $k_\lambda \in \mathbb{Q}(q)^\times$. Multiplying by a power of $(q-1)$, we may further assume that $k_\lambda \in \mathcal{A}_1$ for all $\lambda \in \mathrm{L}$. Set $n := \max\{\, m \geq 0 \,|\, k_\lambda/(q-1)^m \in \mathcal{A}_1$ for all $\lambda \in \mathrm{L} \,\}$. Then there exists an element $\lambda_* \in \mathrm{L}$ such that $k_{\lambda_*}/(q-1)^n \in \mathcal{A}_1 \setminus (q-1)\mathcal{A}_1$. Hence the equality $\sum_{\lambda \in \mathrm{L}} \overline{k_\lambda/(q-1)^n}\, \overline{x_\lambda} = 0$ holds in ${}_1W$ with $\overline{k_{\lambda_*}/(q-1)^n} \neq 0$. This contradicts to the assumption.

Let us prove the second assertion. Let w be an arbitrary non-zero element of W. Then $w = \sum_{\lambda \in \mathrm{M}} c_\lambda x_\lambda$ for some finite subset $\mathrm{M} \subset \Lambda$ with $c_\lambda \in \mathbb{Q}(q)^\times$. Now we set $p := \min\{\, m \geq 0 \,|\, c_\lambda(q-1)^m \in \mathcal{A}_1$ for all $\lambda \in \mathrm{M} \,\}$. We now assume that $p > 0$. Then there exists an element $\lambda_\sharp \in \mathrm{M}$ such that $c_{\lambda_\sharp}(q-1)^p \in \mathcal{A}_1 \setminus (q-1)\mathcal{A}_1$. Hence the equality $0 = \sum_{\lambda \in \mathrm{L}} \overline{c_\lambda(q-1)^p}\, \overline{x_\lambda}$ holds in ${}_1W$ with $\overline{c_{\lambda_\sharp}(q-1)^p} \neq 0$. This contradicts to the assumption. Thus we get $p = 0$. Therefore, all c_λ with $\lambda \in \mathrm{M}$ are non-zero elements of $\mathcal{A}_1$. □

Definition 4.4. For each $\boldsymbol{s} \in \mathcal{W}^*$ and $p \in \mathbb{N}_{\ell(\boldsymbol{s})}$, we define a weight vector $E_{\boldsymbol{s}}(p)$ of U^+ with weight $\phi_{\boldsymbol{s}}(p)$ by setting

$$E_{\boldsymbol{s}}(p) := T_{\boldsymbol{s}(1)} T_{\boldsymbol{s}(2)} \cdots T_{\boldsymbol{s}(p-1)}(E_{\boldsymbol{s}(p)}). \tag{4.32}$$

If $\phi_{\boldsymbol{s}}(p) = \beta$, we denote the $E_{\boldsymbol{s}}(p)$ by $E_{\beta, \boldsymbol{s}}$.

Lemma 4.5. (1) *Let β be an element of Δ^{re}_+, and $\boldsymbol{s}$ an element of $\mathcal{W}^*$ such that $\beta = \phi_{\boldsymbol{s}}(p)$ for some $p \in \mathbb{N}_{\ell(\boldsymbol{s})}$. Then $E_{\beta, \boldsymbol{s}}$ belongs to ${}_{\mathcal{A}_1}U^+_\beta \setminus (q-1){}_{\mathcal{A}_1}U^+$. In particular, the image $\overline{E_{\beta, \boldsymbol{s}}}$ of $E_{\beta, \boldsymbol{s}}$ by the specialization at $q = 1$ is a non-zero element of ${}_1U^+_\beta$.*

(2) *Let* β *be an element of* Δ_+^{re}, *and* x *an element of* W *such that* $\beta \in \Phi(x)$. *We assume that* $E_{\beta,\boldsymbol{s}_1} \in \mathbb{Q}(q)^\times E_{\beta,\boldsymbol{s}_2}$ *for all* $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathcal{W}$ *satisfying* $[\boldsymbol{s}_1] = [\boldsymbol{s}_2] = x$. *Then* $E_{\beta,\boldsymbol{s}_1} = E_{\beta,\boldsymbol{s}_2}$ *for all* $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathcal{W}$ *satisfying* $[\boldsymbol{s}_1] = [\boldsymbol{s}_2] = x$.

(3) *Let* β *be an element of* Δ_+^{re}, *and* x *an element of* W *such that* $\beta \in \Phi(x)$. *We assume that if* $\beta = \sum_{\gamma\in\Phi(x)} \mathbf{c}(\gamma)\gamma$ *with* $\mathbf{c}(\gamma) \in \mathbb{Z}_+$ *for all* $\gamma \in \Phi(x)$ *then* $\mathbf{c}(\beta) = 1$ *and* $\mathbf{c}(\gamma) = 0$ *for all* $\gamma \neq \beta$. *Then* $E_{\beta,\boldsymbol{s}_1} = E_{\beta,\boldsymbol{s}_2}$ *for all* $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathcal{W}$ *satisfying* $[\boldsymbol{s}_1] = [\boldsymbol{s}_2] = x$.

Proof. Let us prove the part (1). By (4.18)–(4.20) and the right equation in (4.21), it is easy to see that $T_i({}_{\mathcal{A}_1}U') = {}_{\mathcal{A}_1}U'$, and hence $T_i((q-1)_{\mathcal{A}_1}U') = (q-1)_{\mathcal{A}_1}U'$ for all $i \in \mathbf{I}$. Thus we see that $T_i({}_{\mathcal{A}_1}U' \setminus (q-1)_{\mathcal{A}_1}U') = {}_{\mathcal{A}_1}U' \setminus (q-1)_{\mathcal{A}_1}U'$ for all $i \in \mathbf{I}$. Then it follows from (4.32) that $E_{\beta,\boldsymbol{s}} = E_{\boldsymbol{s}}(p) \in {}_{\mathcal{A}_1}U^+_\beta \setminus (q-1)_{\mathcal{A}_1}U^+$, since $E_{\boldsymbol{s}(k(p))} \in {}_{\mathcal{A}_1}U^+ \setminus (q-1)_{\mathcal{A}_1}U^+$ and $E_{\beta,\boldsymbol{s}} \in U^+_\beta$.

Let us prove the part (2). Put $l = \ell(x)$. Let (p_1, p_2) be a unique pair of elements of $\mathbb{N}_l$ such that $\phi_{\boldsymbol{s}_i}(p_i) = \beta$ for $i = 1, 2$. To prove the assertion, it suffices to show the equality $E_{\boldsymbol{s}_1}(p_1) = E_{\boldsymbol{s}_2}(p_2)$. Since $\boldsymbol{s}_1$ can be transformed to $\boldsymbol{s}_2$ by a finite sequence of braid relations, we may assume that $\boldsymbol{s}_1$ can be transformed to $\boldsymbol{s}_2$ by the following manner (i) or (ii) or (iii) or (iv):

(i) replacing two consecutive entries s_i, s_j in $\boldsymbol{s}_1$ satisfying $\mathrm{A}_{ij}\mathrm{A}_{ji} = 0$ by s_j, s_i;

(ii) replacing three consecutive entries s_i, s_j, s_i in $\boldsymbol{s}_1$ satisfying $\mathrm{A}_{ij}\mathrm{A}_{ji} = 1$ by s_j, s_i, s_j;

(iii) replacing four consecutive entries s_i, s_j, s_i, s_j in $\boldsymbol{s}_1$ satisfying $\mathrm{A}_{ij}\mathrm{A}_{ji} = 2$ by s_j, s_i, s_j, s_i;

(iv) replacing six consecutive entries $s_i, s_j, s_i, s_j, s_i, s_j$ in $\boldsymbol{s}_1$ satisfying $\mathrm{A}_{ij}\mathrm{A}_{ji} = 3$ by $s_j, s_i, s_j, s_i, s_j, s_i$.

In the case (i), there exists a unique $m_0 \in \mathbb{N}$ such that $\boldsymbol{s}_1(m_0) = s_i$, $\boldsymbol{s}_1(m_0+1) = s_j$, $\boldsymbol{s}_2(m_0) = s_j$, $\boldsymbol{s}_2(m_0+1) = s_i$, and $\boldsymbol{s}_1(m) = \boldsymbol{s}_2(m)$ for all $m \neq m_0, m_0+1$. Suppose that $p_1 < p_2$. Then $p_1 = m_0$ and $p_2 = m_0 + 1$ since $\phi_{\boldsymbol{s}_1}(p_1) = \phi_{\boldsymbol{s}_2}(p_2)$. Thus we get $E_{\boldsymbol{s}_1}(p_1) = E_{\boldsymbol{s}_2}(p_2)$ since $E_i = T_j(E_i)$. Suppose that $p_1 = p_2$. Then $p_1 = p_2 < m_0$ or $m_0 + 1 < p_1 = p_2$ since $\phi_{\boldsymbol{s}_1}(p_1) = \phi_{\boldsymbol{s}_2}(p_2)$, hence the equality is valid since $T_iT_j = T_jT_i$ and $\boldsymbol{s}_1(m) = \boldsymbol{s}_2(m)$ for all $m \neq m_0, m_0+1$.

In the case (ii), there exists a unique $m_0 \in \mathbb{N}$ such that $\boldsymbol{s}_1(m_0) = s_i$, $\boldsymbol{s}_1(m_0+1) = s_j$, $\boldsymbol{s}_1(m_0+2) = s_i$, $\boldsymbol{s}_2(m_0) = s_j$, $\boldsymbol{s}_2(m_0+1) = s_i$, $\boldsymbol{s}_2(m_0+2) = s_j$ and $\boldsymbol{s}_1(m) = \boldsymbol{s}_2(m)$ for all $m \neq m_0, m_0+1, m_0+2$. Suppose that $p_1 < p_2$. Then $p_1 = m_0$ and $p_2 = m_0 + 2$, since $\phi_{\boldsymbol{s}_1}(p_1) = \phi_{\boldsymbol{s}_2}(p_2)$. Thus we get $E_{\boldsymbol{s}_1}(p_1) = E_{\boldsymbol{s}_2}(p_2)$, since $E_i = T_jT_i(E_j)$. Suppose that $p_1 = p_2$. Then there exist three cases (a)–(c) to be considered: (a) $p_1 = p_2 < m_0$, (b) $m_0 + 2 < p_1 = p_2$, (c) $p_1 = p_2 = m_0 + 1$, since $\phi_{\boldsymbol{s}_1}(p_1) = \phi_{\boldsymbol{s}_2}(p_2)$. In the case (a) or (b), the equality is valid since $T_iT_jT_i = T_jT_iT_j$ and $\boldsymbol{s}_1(m) = \boldsymbol{s}_2(m)$ for all $m \neq m_0, m_0+1, m_0+2$. In the case (c), $E_{\boldsymbol{s}_1}(p_1)$ and $E_{\boldsymbol{s}_2}(p_2)$ are not proportional since $T_i(E_j)$ and $T_j(E_i)$ are not proportional, which contradicts the assumption of (2). Therefore the assertion is valid in the case (ii).

Since the arguments for the cases (iii) and (iv) are similar to that for the case (ii), we omit them.

Let us prove the part (3). Put $l = \ell(x)$. Let (p_1, p_2) be a unique pair of elements of $\mathbb{N}_l$ such that $\phi_{\boldsymbol{s}_i}(p_i) = \beta$ for $i = 1, 2$. Then $E_{\beta,\boldsymbol{s}_i} = E_{\boldsymbol{s}_i}(p_i)$ for $i = 1, 2$. By Proposition 40.2.1 in [14], we see that

$$E_{\boldsymbol{s}_1}(p_1) = \sum_{(c_1,c_2,\cdots,c_l)\in(\mathbb{Z}_+)^l} k_{(c_1,c_2,\cdots,c_l)} E_{\boldsymbol{s}_2}(1)^{c_1} E_{\boldsymbol{s}_2}(2)^{c_2} \cdots E_{\boldsymbol{s}_2}(l)^{c_l},$$

where $k_{(c_1,c_2,\cdots,c_l)} \in \mathbb{Q}(q)$. Now suppose that $(c_1, c_2, \cdots, c_l)$ is a sequence such that $k_{(c_1,c_2,\cdots,c_l)} \neq 0$. Then $\sum_{p=1}^{l} c_p \phi_{\boldsymbol{s}_2}(p) = \beta$. By the assumption, we see that $c_{p_2} = 1$ and $c_p = 0$ for all $p \neq p_2$. Thus $E_{\boldsymbol{s}_1}(p_1) = kE_{\boldsymbol{s}_2}(p_2)$ for some $k \in \mathbb{Q}(q)^{\times}$. By the part (2), we get $E_{\boldsymbol{s}_1}(p_1) = E_{\boldsymbol{s}_2}(p_2)$, i.e., $E_{\beta,\boldsymbol{s}_1} = E_{\beta,\boldsymbol{s}_2}$. □

5. The subalgebra $U_{\mathbf{J}}$ associated with $\Delta_{\mathbf{J}}$ and the braid group action

Throughout this section, we assume that $\mathfrak{g}$ is the affine Kac-Moody Lie algebra of the type $X_r^{(1)}$ $(X = A, B, C, D, E, F, G)$ with Δ the root system.

Lemma 5.1. *Let ε be an element of $\mathring{\Delta}_+$. If $(\boldsymbol{s}_1, \boldsymbol{s}_2)$ is a pair of elements of $\mathcal{W}$ such that $\delta - \varepsilon \in \Phi([\boldsymbol{s}_i]) \subset \Delta(1, -)$ for $i = 1, 2$, then $E_{\delta-\varepsilon,\boldsymbol{s}_1} = E_{\delta-\varepsilon,\boldsymbol{s}_2}$.*

Proof. We may assume that $[\boldsymbol{s}_1] = [\boldsymbol{s}_2]$, and put $x = [\boldsymbol{s}_1] = [\boldsymbol{s}_2]$. Since $\gamma \in \Delta(1, -)$ for each $\gamma \in \Phi(x)$, there exists $\mathbf{d}(\gamma) \in \mathbb{N}$ such that $\gamma = \mathbf{d}(\gamma)\delta + \overline{\gamma}$ with $\overline{\gamma} \in \mathring{\Delta}_-$. Now suppose that $\delta - \varepsilon = \sum_{\gamma\in\Phi(x)} \mathbf{c}(\gamma)\gamma$ with $\mathbf{c}(\gamma) \in \mathbb{Z}_+$ for all $\gamma \in \Phi(x)$. Then $\delta - \varepsilon = (\sum_{\gamma\in\Phi(x)} \mathbf{c}(\gamma)\mathbf{d}(\gamma))\delta + \sum_{\gamma\in\Phi(x)} \mathbf{c}(\gamma)\overline{\gamma}$, which implies that $\mathbf{c}(\delta - \varepsilon) = \mathbf{d}(\delta - \varepsilon) = 1$ and $\mathbf{c}(\gamma) = 0$ for all $\gamma \neq \delta - \varepsilon$. Thus the Lemma follows immediately from Lemma 4.5(3). □

Definition 5.2. For each $\varepsilon \in \mathring{\Delta}_+$, we define a weight vector $E_{\delta-\varepsilon}$ of U^+ with weight $\delta - \varepsilon$ by setting

$$E_{\delta-\varepsilon} := E_{\delta-\varepsilon,\boldsymbol{s}}, \tag{5.1}$$

where $\boldsymbol{s}$ is an element of $\mathcal{W}$ such that $\delta - \varepsilon \in \Phi([\boldsymbol{s}]) \subset \Delta(1, -)$. By Lemma 5.1, we see that the vector $E_{\delta-\varepsilon}$ is independent from the choice of $\boldsymbol{s}$.

Definition 5.3. For each non-empty subset $\mathbf{J} \subset \mathring{\mathbf{I}}$, we define subalgebras of U over $\mathbb{Q}(q)$ by setting

$$U_{\mathbf{J}} := \langle\, E_\alpha, K_\alpha^{\pm 1}, F_\alpha \mid \alpha \in \Pi_{\mathbf{J}} \,\rangle_{\mathbb{Q}(q)\text{-alg}}, \quad U_{\mathbf{J}}^0 := \langle\, K_\alpha^{\pm 1} \mid \alpha \in \Pi_{\mathbf{J}} \,\rangle_{\mathbb{Q}(q)\text{-alg}},$$
$$U_{\mathbf{J}}^+ := \langle\, E_\alpha \mid \alpha \in \Pi_{\mathbf{J}} \,\rangle_{\mathbb{Q}(q)\text{-alg}}, \quad U_{\mathbf{J}}^{\geq 0} := \langle\, E_\alpha, K_\alpha^{\pm 1} \mid \alpha \in \Pi_{\mathbf{J}} \,\rangle_{\mathbb{Q}(q)\text{-alg}},$$
$$U_{\mathbf{J}}^- := \langle\, F_\alpha \mid \alpha \in \Pi_{\mathbf{J}} \,\rangle_{\mathbb{Q}(q)\text{-alg}}, \quad U_{\mathbf{J}}^{\leq 0} := \langle\, K_\alpha^{\pm 1}, F_\alpha \mid \alpha \in \Pi_{\mathbf{J}} \,\rangle_{\mathbb{Q}(q)\text{-alg}},$$

where $F_\alpha := \Omega(E_\alpha)$. Note that $U_{\mathbf{J}}^0 = \{K_\alpha \,|\, \alpha \in Q_{\mathbf{J}}\}$ and that if $\mathbf{J} = \mathring{\mathbf{I}}$ then $U_{\mathbf{J}} = U'$ and $U_{\mathbf{J}}^{\pm} = U^{\pm}$. For each $\mu \in Q_{\mathbf{J}+}$, we set $U_{\mathbf{J}\mu}^+ := U_{\mathbf{J}}^+ \cap U_\mu^+$. Note that

$$U_{\mathbf{J}}^+ = \oplus_{\mu\in Q_{\mathbf{J}+}} U_{\mathbf{J}\mu}^+.$$

Define $\tilde{\Delta}_{\mathbf{J}+}^{im} \subset \tilde{\Delta}_+^{im}$ and $\tilde{\Delta}_{\mathbf{J}+} \subset \tilde{\Delta}_+$ by setting

$$\tilde{\Delta}_{\mathbf{J}+}^{im} := \{\, (m\delta, j) \mid m \in \mathbb{N},\, j \in \mathbf{J} \,\}, \qquad \tilde{\Delta}_{\mathbf{J}+} := \Delta_{\mathbf{J}+}^{re} \amalg \tilde{\Delta}_{\mathbf{J}+}^{im}.$$

and define $\kappa_{\mathbf{J}} \colon Q_{\mathbf{J}+} \to \mathbb{N}$ by setting for each $\mu \in Q_{\mathbf{J}+}$,

$$\kappa_{\mathbf{J}}(\mu) := \sharp\{\, \mathbf{c} \colon \tilde{\Delta}_{\mathbf{J}+} \to \mathbb{Z}_+ \mid \sum_{\alpha\in\Delta_{\mathbf{J}+}^{re}} \mathbf{c}(\alpha)\alpha + \sum_{m\delta\in\Delta_+^{im}} \sum_{j\in\mathbf{J}} \mathbf{c}((m\delta, j))m\delta = \mu \,\}.$$

Lemma 5.4. *Let $\mathbf{J}$ and $\mathbf{J}'$ be connected subsets of $\mathring{\mathbf{I}}$ which are disjointed with each other, and j an arbitrary element of $\mathbf{J}_*$.*

(1) *Let j^- be a unique element of $\mathbf{J}_*$ such that $\rho_{\mathbf{J}j}(\alpha_{j^-}) = \delta - \theta_{\mathbf{J}}$. Then*

$$E_{\delta-\theta_{\mathbf{J}}} = T_{(\varepsilon_j)^{\mathbf{J}}}(E_{j^-}), \quad K_{\delta-\theta_{\mathbf{J}}} = T_{(\varepsilon_j)^{\mathbf{J}}}(K_{j^-}), \quad F_{\delta-\theta_{\mathbf{J}}} = T_{(\varepsilon_j)^{\mathbf{J}}}(F_{j^-}). \tag{5.2}$$

Here the translation t_{ε_j} is simply denoted by ε_j. Let $w_\circ$ and $w_{\circ j}$ be the longest element of $\mathring{W}_{\mathbf{J}}$ and $\mathring{W}_{\mathbf{J}\setminus\{j\}}$, respectively, and set $w_{\mathbf{J}j} := w_\circ w_{\circ j}$. Then we have

$$T_{(\varepsilon_j)^{\mathbf{J}}} = T_{\varepsilon_j} T_{w_{\mathbf{J}j}}^{-1}. \tag{5.3}$$

In particular, we have

$$E_{\delta-\alpha_j} = T_{\varepsilon_j} T_j^{-1}(E_j), \quad K_{\delta-\alpha_j} = T_{\varepsilon_j} T_j^{-1}(K_j), \quad F_{\delta-\alpha_j} = T_{\varepsilon_j} T_j^{-1}(F_j). \tag{5.4}$$

(2) *For each $i \in \mathring{\mathbf{I}} \setminus \mathbf{J}$, $z \in \mathring{W}_{\mathbf{J}'}$, and $j' \in \mathbf{J}'_*$, we have*

$$\text{(i)}\ [T_{(\varepsilon_j)^{\mathbf{J}}}, T_{\varepsilon_i}] = 0, \quad \text{(ii)}\ [T_{(\varepsilon_j)^{\mathbf{J}}}, T_z] = 0, \quad \text{(iii)}\ [T_{(\varepsilon_j)^{\mathbf{J}}}, T_{(\varepsilon_{j'})^{\mathbf{J}'}}] = 0. \tag{5.5}$$

(3) *For each $i \in \mathring{\mathbf{I}} \setminus \mathbf{J}$, $z \in \mathring{W}_{\mathbf{J}'}$, and $(X, Y) \in U_{\mathbf{J}} \times U_{\mathbf{J}'}$, we have*

$$\text{(i)}\ T_{\varepsilon_i}(X) = X, \quad \text{(ii)}\ T_z(X) = X, \quad \text{(iii)}\ T_{(\varepsilon_j)^{\mathbf{J}}}(Y) = Y. \tag{5.6}$$

(4) *For each $(X, Y) \in U_{\mathbf{J}} \times U_{\mathbf{J}'}$, we have $[X, Y] = 0$.*

Proof. (1) By Lemma 2.7(2.5)(i), (2.13)(i), and Definition 5.2, we have

$$E_{\delta-\theta_{\mathbf{J}}} = T_{(\varepsilon_j)^{\mathbf{J}}}(E_{j^-}), \quad K_{\delta-\theta_{\mathbf{J}}} = T_{(\varepsilon_j)^{\mathbf{J}}}(K_{j^-}).$$

Since $\Omega T_{(\varepsilon_j)^{\mathbf{J}}} = T_{(\varepsilon_j)^{\mathbf{J}}}\Omega$, we have

$$F_{\delta-\theta_{\mathbf{J}}} = \Omega(E_{\delta-\theta_{\mathbf{J}}}) = T_{(\varepsilon_j)^{\mathbf{J}}}(F_{j^-}).$$

The (5.3) follows from the following equalities:

$$\ell((\varepsilon_j)^{\mathbf{J}}) + \ell((\varepsilon_j)_{\mathbf{J}}) = \ell(\varepsilon_j), \quad (\varepsilon_j)_{\mathbf{J}} = w_{\mathbf{J}j}.$$

In the case where $\mathbf{J} = \{j\}$, we have $j^- = j$ and $w_{\mathbf{J}j} = s_j$. Hence we have (5.4) by (5.3).

The part (2) follows from Lemma 2.7(2).

(3) Since $t_{\varepsilon_i}(\alpha) = \alpha$ we have $T_{\varepsilon_i}(E_\alpha) = E_\alpha$, $T_{\varepsilon_i}(K_\alpha) = K_\alpha$, and $T_{\varepsilon_i}(F_\alpha) = F_\alpha$ for each $\alpha \in \mathring{\Pi}_{\mathbf{J}}$. By (5.2) and (5.5)(i), we see that $T_{\varepsilon_i}(E_{\delta-\theta_{\mathbf{J}}}) = E_{\delta-\theta_{\mathbf{J}}}$, $T_{\varepsilon_i}(K_{\delta-\theta_{\mathbf{J}}}) = K_{\delta-\theta_{\mathbf{J}}}$, and $T_{\varepsilon_i}(F_{\delta-\theta_{\mathbf{J}}}) = F_{\delta-\theta_{\mathbf{J}}}$, and hence we have (i). Since $z(\alpha) = \alpha$ we have $T_z(E_\alpha) = E_\alpha$, $T_z(K_\alpha) = K_\alpha$, and $T_z(F_\alpha) = F_\alpha$ for each $\alpha \in \mathring{\Pi}_{\mathbf{J}}$. By (5.2) and (5.5)(ii), we see that $T_z(E_{\delta-\theta_{\mathbf{J}}}) = E_{\delta-\theta_{\mathbf{J}}}$, $T_z(K_{\delta-\theta_{\mathbf{J}}}) = K_{\delta-\theta_{\mathbf{J}}}$, and $T_z(F_{\delta-\theta_{\mathbf{J}}}) = F_{\delta-\theta_{\mathbf{J}}}$, and hence we have (ii). The (iii) follows from (i)(ii) and (5.3).

(4) It suffices to prove that $[X_\alpha, Y_{\alpha'}] = 0$ for $X, Y = E, K, F$ and $(\alpha, \alpha') \in \Pi_{\mathbf{J}} \times \Pi_{\mathbf{J}'}$. Since $(\alpha \,|\, \alpha') = 0$, it is clear that $[E_\alpha, K_{\alpha'}] = [F_\alpha, K_{\alpha'}] = 0$. Let us prove that $[E_\alpha, E_{\alpha'}] = 0$. Suppose that $j \in \mathbf{J}_*$ and $j' \in \mathbf{J}'_*$. In the case where $(\alpha, \alpha') \in \mathring{\Pi}_{\mathbf{J}} \times \mathring{\Pi}_{\mathbf{J}'}$, it is clear that $[E_\alpha, E_{\alpha'}] = 0$. In the case where $\alpha = \delta - \theta_{\mathbf{J}}$ and $\alpha' \in \mathring{\Pi}_{\mathbf{J}'}$, we have

$$[E_\alpha, E_{\alpha'}] = T_{(\varepsilon_j)^{\mathbf{J}}}([E_{j_*}, E_{\alpha'}]) = 0$$

by (5.2) and (3), where j^- is a unique element of $\mathbf{J}$ such that $\rho_{\mathbf{J}j}(\alpha_{j^-}) = \alpha$. In the case where $\alpha = \delta - \theta_{\mathbf{J}}$ and $\alpha' = \delta - \theta_{\mathbf{J}'}$, we have

$$[E_\alpha, E_{\alpha'}] = T_{(\varepsilon_j)^{\mathbf{J}}} T_{(\varepsilon_{j'})^{\mathbf{J}'}}([E_{j^-}, E_{j'^-}]) = 0$$

by (5.2) and (3), where j'^- is a unique element of $\mathbf{J}'_*$ such that $\rho_{\mathbf{J}'j'}(\alpha_{j'^-}) = \alpha'$. Similarly, we can prove that $[E_\alpha, F_{\alpha'}] = [F_\alpha, F_{\alpha'}] = 0$. □

Proposition 5.5. *Let* $\mathbf{J}$ *be a non-empty subset of* $\overset{\circ}{\mathbf{I}}$*, and* $\mathbf{J}_1, \dots, \mathbf{J}_{\mathrm{C}(\mathbf{J})}$ *the connected components of* $\mathbf{J}$ *with* $\mathrm{C}(\mathbf{J})$ *the number of the connected components. If* $\mathbf{J}_c$ *and* $\mathbf{J}_{c'}$ *are different connected components of* $\mathbf{J}$*, then* $[X, X'] = 0$ *for all* $(X, X') \in U_{\mathbf{J}_c} \times U_{\mathbf{J}_{c'}}$*. Moreover, the following equality holds*:

$$U_{\mathbf{J}} = \mathrm{span}_{\mathbb{Q}(q)}\{ \textstyle\prod_{c=1}^{\mathrm{C}(\mathbf{J})} X_c \mid X_c \in U_{\mathbf{J}_c} \}. \tag{5.7}$$

Proof. The first assertion follows from Lemma 5.4(4), and the second assertion follows from the first assertion and Definition 5.3. □

Proposition 5.6. (1) *Let us use the notation introduced in Definition 3.4. For each* $\rho \in \Omega_{\mathbf{J}}$ *and* $\alpha \in \Pi_{\mathbf{J}}$*, the following equalities hold*:

$$T_{\widetilde{\rho}}(E_\alpha) = E_{\rho(\alpha)}, \qquad T_{\widetilde{\rho}}(K_\alpha) = K_{\rho(\alpha)}, \qquad T_{\widetilde{\rho}}(F_\alpha) = F_{\rho(\alpha)}. \tag{5.8}$$

In particular, the restriction $T_{\widetilde{\rho}}|_{U_{\mathbf{J}}}$ *is an automorphism of* $U_{\mathbf{J}}$*.*

(2) *Let* $\mathbf{J}$ *and* $\mathbf{J}'$ *be connected subsets of* $\overset{\circ}{\mathbf{I}}$ *which are disjointed with each other. Then* $[T_{\widetilde{\tau}}, T_{\widetilde{\sigma}}] = 0$ *for all* $(\tau, \sigma) \in \Omega_{\mathbf{J}} \times \Omega_{\mathbf{J}'}$*. Moreover,* $T_{\widetilde{\tau}}(X) = X$ *for all* $\tau \in \Omega_{\mathbf{J}}$ *and* $X \in U_{\mathbf{J}'}$*.*

Proof. (1) By Proposition 2.5, we may assume that $\rho = \rho_{\mathbf{J}j}$ with $j \in \mathbf{J}_*$. Then we have $T_{\widetilde{\rho}} = T_{(\varepsilon_j)^{\mathbf{J}}}$ by Definition 3.5. By (2.10) and (2.12), we have $\ell(\{(\varepsilon_j)^{\mathbf{J}}\}^2) = 2\ell((\varepsilon_j)^{\mathbf{J}})$, and hence $T_{\{(\varepsilon_j)^{\mathbf{J}}\}^2} = (T_{(\varepsilon_j)^{\mathbf{J}}})^2$. In the case where $\alpha = \delta - \theta_{\mathbf{J}}$, by (5.2) we have

$$T_{\widetilde{\rho}}(E_\alpha) = (T_{(\varepsilon_j)^{\mathbf{J}}})^2(E_{j^-}) = T_{\{(\varepsilon_j)^{\mathbf{J}}\}^2}(E_{j^-}) = E_{\rho(\alpha)}$$

since

$$\rho(\alpha) = \rho(\delta - \theta_{\mathbf{J}}) = \{(\varepsilon_j)^{\mathbf{J}}\}^2(\alpha_{j^-}) = \alpha_j \in \overset{\circ}{\Pi}_{\mathbf{J}}.$$

In the case where $\alpha = \alpha_{j^-}$, the equalities in (5.8) are nothing but the equalities in (5.2). In the case where $\alpha \in \overset{\circ}{\Pi}_{\mathbf{J}} \setminus \{\alpha_{j^-}\}$, since $\rho(\alpha) \in \overset{\circ}{\Pi}_{\mathbf{J}}$, the equalities in (5.8) is clear.

(2) The first assertion follows from (5.5)(iii), and the second assertion follows from (5.6)(iii). □

Proposition 5.7. (1) *Let* $\mathbf{J}$ *be an arbitrary connected subset of* $\overset{\circ}{\mathbf{I}}$*. Then the* $\mathbb{Q}(q)$*-subalgebra* $U_{\mathbf{J}}$ *of* U *is characterized as the associative* $\mathbb{Q}(q)$*-algebra with the unit* 1 *defined by the generators* $\{E_\alpha, K_\alpha^{\pm 1}, F_\alpha \,|\, \alpha \in \Pi_{\mathbf{J}}\}$ *and the following fundamental relations*:

$$[K_\alpha\,,\, K_\beta] = 0, \quad K_\alpha K_\alpha^{-1} = K_\alpha^{-1} K_\alpha = 1, \tag{5.9}$$

$$K_\alpha E_\beta K_\alpha^{-1} = q^{(\beta|\alpha)} E_\beta, \quad K_\alpha F_\beta K_\alpha^{-1} = q^{-(\beta|\alpha)} F_\beta, \tag{5.10}$$

$$[E_\alpha\,,\, F_\beta] = \delta_{\alpha\beta}(K_\alpha - K_\alpha^{-1})/(q_\alpha - q_\alpha^{-1}), \tag{5.11}$$

$$(ad_q E_\alpha)^{(1-\mathrm{A}_{\alpha\beta})}.E_\beta = (ad_q F_\alpha)^{(1-\mathrm{A}_{\alpha\beta})}.F_\beta = 0 \quad (\alpha \neq \beta), \tag{5.12}$$

where $\alpha, \beta \in \Pi_{\mathbf{J}}$*. Moreover, the following equality holds*:

$$U^+_{\mathbf{J}} = \oplus_{\mu \in Q_{\mathbf{J}+}} U^+_{\mathbf{J}\mu} \tag{5.13}$$

with the following equality:

$$\dim_{\mathbb{Q}(q)} U^+_{\mathbf{J}\mu} = \kappa_{\mathbf{J}}(\mu). \tag{5.14}$$

(2) *For each non-empty subset* $\mathbf{J} \subset \overset{\circ}{\mathbf{I}}$*, the multiplication defines the following isomorphism* m *of* $\mathbb{Q}(q)$*-vector spaces*:

$$m\colon U_{\mathbf{J}}^{+} \otimes U_{\mathbf{J}}^{0} \otimes U_{\mathbf{J}}^{-} \xrightarrow{\sim} U_{\mathbf{J}}. \tag{5.15}$$

Proof. It is clear that all of the clams in (1) and (2) are valid in the case where $\mathbf{J} = \overset{\circ}{\mathbf{I}}$. Hence we may assume that $\mathbf{J}$ is a non-empty proper subset of $\overset{\circ}{\mathbf{I}}$. Then we see that the irreducible root system $\overset{\circ}{\Delta}_{\mathbf{J}}$ is not of type E_8 or F_4 or G_2, and hence $\sharp\mathbf{J}_* \geq 1$.

Let $\check{U}_{\mathbf{J}}$ be the associative $\mathbb{Q}(q)$-algebra with the unit 1 defined by the generators $\{\check{E}_\alpha, \check{F}_\alpha, \check{K}_\alpha^{\pm 1} \mid \alpha \in \Pi_{\mathbf{J}}\}$ and the fundamental relations (5.9)–(5.12) with replacing X_α by $\check{X}_\alpha$ for $X = E, K^{\pm 1}, F$ with $\alpha \in \Pi_{\mathbf{J}}$. To prove the part (1), it suffices to prove the claim that the assignment $\check{X}_\alpha \mapsto X_\alpha$ for $X = E, K^{\pm 1}, F$ with $\alpha \in \Pi_{\mathbf{J}}$ defines a $\mathbb{Q}(q)$-algebra isomorphism $h_{\mathbf{J}}\colon \check{U}_{\mathbf{J}} \to U_{\mathbf{J}}$. In the case where $\sharp\mathbf{J} = 1$, the claim is nothing but that of Proposition 3.8 in [1]. Hence we may assume that $\sharp\mathbf{J} \geq 2$. To prove the well-definedness of $h_{\mathbf{J}}$, we show that the generators $\{E_\alpha, K_\alpha^{\pm 1}, F_\alpha \mid \alpha \in \Pi_{\mathbf{J}}\}$ of $U_{\mathbf{J}}$ satisfies the relations (5.9)–(5.12). The relations (5.9), (5.10), and (5.11) for $\alpha = \beta$ are clear. Thus it suffices to prove the relations (5.11) for $\alpha \neq \beta$ and (5.12). In the case where $\{\alpha, \beta\} \subset \overset{\circ}{\Pi}_{\mathbf{J}}$, the relations (5.11) for $\alpha \neq \beta$ and (5.12) are clear.

Suppose that $\{\alpha, \beta\} = \{\alpha_j, \delta - \theta_{\mathbf{J}}\}$ with $j \in \mathbf{J}_*$ satisfying $\mathrm{ord}(\rho_{\mathbf{J}j}) \geq 3$. Then, there exists an element τ of the cyclic group generated by $\rho_{\mathbf{J}j}$ such that $\tau(\alpha)$ and $\tau(\beta)$ are distinct elements of $\overset{\circ}{\Pi}_{\mathbf{J}}$. Since $\mathrm{A}_{\alpha\beta} = \mathrm{A}_{\tau(\alpha)\tau(\beta)}$, it follows from Proposition 5.6 that

$$T_{\tilde{\tau}}([E_\alpha, F_\beta]) = [E_{\tau(\alpha)}, F_{\tau(\beta)}] = 0, \tag{5.16}$$

$$T_{\tilde{\tau}}((ad_q E_\alpha)^{(1-\mathrm{A}_{\alpha\beta})}.E_\beta) = (ad_q E_{\tau(\alpha)})^{(1-\mathrm{A}_{\tau(\alpha)\tau(\beta)})}.E_{\tau(\beta)} = 0, \tag{5.17}$$

$$T_{\tilde{\tau}}((ad_q F_\alpha)^{(1-\mathrm{A}_{\alpha\beta})}.F_\beta) = (ad_q F_{\tau(\alpha)})^{(1-\mathrm{A}_{\tau(\alpha)\tau(\beta)})}.F_{\tau(\beta)} = 0. \tag{5.18}$$

Since $T_{\tilde{\tau}}$ is an automorphism of $U_{\mathbf{J}}$, the equalities (5.16)(5.17)(5.18) imply the relations (5.11) for $\alpha \neq \beta$ and (5.12) are valid in this case.

Suppose that $\{\alpha, \beta\} = \{\alpha_j, \delta - \theta_{\mathbf{J}}\}$ with $j \in \mathbf{J}_*$ satisfying $(\rho_{\mathbf{J}j})^2 = 1$. By Lemma 2.8 and Definition 5.2, we see that $E_\alpha = T_z(E_i)$, $E_\beta = T_z(E_{i'})$, and $F_\beta = T_z(F_{i'})$ for some $z \in W(\Delta(1,-))$ and distinct elements $i, i' \in \mathbf{I}$. Since $\mathrm{A}_{\alpha\beta} = \mathrm{A}_{ii'}$, it follows that

$$[E_\alpha, F_\beta] = T_z([E_i, F_{i'}]) = 0,$$

$$(ad_q E_\alpha)^{(1-\mathrm{A}_{\alpha\beta})}.E_\beta = T_z((ad_q E_i)^{(1-\mathrm{A}_{ii'})}.E_{i'}) = 0,$$

$$(ad_q F_\alpha)^{(1-\mathrm{A}_{\alpha\beta})}.F_\beta = T_z((ad_q F_i)^{(1-\mathrm{A}_{ii'})}.F_{i'}) = 0.$$

Suppose that $\{\alpha, \beta\} = \{\alpha_{j'}, \delta - \theta_{\mathbf{J}}\}$ with $j' \in \mathbf{J} \setminus \mathbf{J}_*$. By Proposition 2.5 and the first assertion of Lemma 2.7(5), we see that $\rho_{\mathbf{J}j}(\alpha)$ and $\rho_{\mathbf{J}j}(\beta)$ are distinct elements of $\overset{\circ}{\Pi}_{\mathbf{J}}$ for each $j \in \mathbf{J}_*$. Put $\rho = \rho_{\mathbf{J}j}$. Since $\mathrm{A}_{\alpha\beta} = \mathrm{A}_{\rho(\alpha)\rho(\beta)}$, it follows from Proposition 5.6 that the equalities (5.16)(5.17)(5.18) hold with replacing τ by ρ. Hence the relations (5.11) for $\alpha \neq \beta$ and (5.12) are valid in this case.

We next prove (2). It is clear that $U_{\mathbf{J}}^{+} \subset U^{+}$, $U_{\mathbf{J}}^{0} \subset U^{0}$, and $U_{\mathbf{J}}^{-} \subset U^{-}$, and hence the multiplication mapping m is an injective $\mathbb{Q}(q)$-linear mapping. In the case where $\mathbf{J}$ is connected, by (5.9)–(5.11), we see that m is surjective. In the general case, the surjectivity of m follows from Lemma 5.4(4).

The surjectivity of $h_{\mathbf{J}}$ is clear. We prove the injectivity of $h_{\mathbf{J}}$. Let $\check{U}_{\mathbf{J}}^{+}$ be the subalgebra of $\check{U}_{\mathbf{J}}$ generated by $\{\check{E}_\alpha \,|\, \alpha \in \Pi_{\mathbf{J}}\}$, $\check{U}_{\mathbf{J}}^{0}$ the subalgebra of $\check{U}_{\mathbf{J}}$ generated by $\{\check{K}_\alpha^{\pm 1} \,|\, \alpha \in \Pi_{\mathbf{J}}\}$, and $\check{U}_{\mathbf{J}}^{-}$ the subalgebra of $\check{U}_{\mathbf{J}}$ generated by $\{\check{F}_\alpha \,|\, \alpha \in \Pi_{\mathbf{J}}\}$. Then it is clear that $h_{\mathbf{J}}(\check{U}_{\mathbf{J}}^{+}) = (U_{\mathbf{J}}^{+})$, $h_{\mathbf{J}}(\check{U}_{\mathbf{J}}^{0}) = (U_{\mathbf{J}}^{0})$, and $h_{\mathbf{J}}(\check{U}_{\mathbf{J}}^{-}) = (U_{\mathbf{J}}^{-})$. Set $h_{\mathbf{J}}^{\pm} := h_{\mathbf{J}}|_{U_{\mathbf{J}}^{\pm}}$ and $h_{\mathbf{J}}^{0} := h_{\mathbf{J}}|_{U_{\mathbf{J}}^{0}}$. Then we see that

$$h_{\mathbf{J}} \circ \check{m} = m \circ (h_{\mathbf{J}}^{+} \otimes h_{\mathbf{J}}^{0} \otimes h_{\mathbf{J}}^{-}),$$

where $\check{m}$ is the multiplication mapping $\check{U}_{\mathbf{J}}^{+} \otimes \check{U}_{\mathbf{J}}^{0} \otimes \check{U}_{\mathbf{J}}^{-} \to \check{U}_{\mathbf{J}}$. Since both m and $\check{m}$ are isomorphisms of $\mathbb{Q}(q)$-vector spaces, it suffices to show that $\check{U}_{\mathbf{J}}^{\pm} \cap \operatorname{Ker} h_{\mathbf{J}} = \{0\}$ and $\check{U}_{\mathbf{J}}^{0} \cap \operatorname{Ker} h_{\mathbf{J}} = \{0\}$. It is clear that $\check{U}_{\mathbf{J}}^{0} \cap \operatorname{Ker} h_{\mathbf{J}} = \{0\}$. Now suppose that $u \in \check{U}_{\mathbf{J}}^{-} \cap \operatorname{Ker} h_{\mathbf{J}}$. Let λ be an element of $\mathfrak{h}^*$ such that $2(\alpha \,|\, \lambda)/(\alpha \,|\, \alpha) = 1$ for all $\alpha \in \Pi_{\mathbf{J}}$. For each $n \in \mathbb{N}$, let $\rho_n \colon U \to \operatorname{End}(M(n\lambda))$ be the representation of U on the Verma module $M(n\lambda)$ with highest weight $n\lambda$, and v_n a highest weight vector of $M(n\lambda)$. Set $M_n := \rho_n(U_{\mathbf{J}})v_n$. Since $\rho_n(U_{\mathbf{J}}^{+})v_n = \{0\}$, we see that $M_n = \rho_n(U_{\mathbf{J}}^{-})v_n$ and $U^0 M_n = M_n$. It follows that $M_n = \oplus_{\alpha \in Q_{\mathbf{J}+}} (M_n \cap M(n\lambda)_{n\lambda - \alpha})$ and $\dim_{\mathbb{Q}(q)}(M_n \cap M(n\lambda)_{n\lambda}) = 1$, where $M(n\lambda)_{n\lambda-\alpha}$ is the weight space of $M(n\lambda)$ with weight $n\lambda - \alpha$. Therefore we may regards the composition $\rho_n \circ h_{\mathbf{J}}$ as a highest weight representation of $\check{U}_{\mathbf{J}}$ on M_n with highest weight $n\lambda$. Hence there exists a unique irreducible quotient L_n of M_n as $\check{U}_{\mathbf{J}}$-module. Since $u \in \check{U}_{\mathbf{J}}^{-} \cap \operatorname{Ker} h_{\mathbf{J}}$, we see that $uL_n = \{0\}$ for all $n \in \mathbb{N}$. By the assumptions of λ, we see that L_n is an integrable highest weight $\check{U}_{\mathbf{J}}$-module for each $n \in \mathbb{N}$. Thus we get $u \in \cap_{n>0} \left(\sum_{\alpha \in \Pi_{\mathbf{J}}} \check{U}_{\mathbf{J}}^{-} \check{F}_\alpha^{n+1}\right)$, and hence $u = 0$. Similarly, we have $\check{U}_{\mathbf{J}}^{+} \cap \operatorname{Ker} h_{\mathbf{J}} = \{0\}$ by considering lowest weight modules.

The (5.13) follows immediately from Definition 5.3, and the (5.14) follows from the characterization of $U_{\mathbf{J}}$ above and (4.31) in Proposition 4.2. □

Remark 5.8. (1) *In the case where* $\sharp\mathbf{J} = 1$*, the characterization of* $U_{\mathbf{J}}$ *described in the part (1) of Proposition 5.7 is given by J. Beck in* [1].

(2) *We will show that the part (1) of Proposition 5.7 is still valid in the case where* $\mathbf{J}$ *is an arbitrary non-empty subset of* $\mathring{\mathbf{I}}$ *(see Proposition 7.1).*

Lemma 5.9. *Let* $\mathbf{J}$ *be an arbitrary connected subset of* $\mathring{\mathbf{I}}$*. Then, for each* $j \in \mathbf{J}$*, the following equality holds*:

$$T_j|_{U_{\mathbf{J}}} = h_{\mathbf{J}} \circ \check{T}_j \circ h_{\mathbf{J}}^{-1}, \tag{5.19}$$

where $h_{\mathbf{J}} \colon \check{U}_{\mathbf{J}} \to U_{\mathbf{J}}$ *is the* $\mathbb{Q}(q)$*-algebra isomorphism introduced in the proof of Proposition 5.7 and* $\check{T}_j$ *is the Lusztig's automorphism of* $\check{U}_{\mathbf{J}}$*.*

Proof. We note that the proof is similar to that of Corollary (a) of Proposition 3.8 in [1]. Let M be an arbitrary integrable $U_q(\mathfrak{g})$-module. Then M can be regarded as an integrable $\check{U}_{\mathbf{J}}$-module via $h_{\mathbf{J}}$. It follows from Lemma 4.1 that

$$\begin{aligned} T_j(E_{\delta-\theta_{\mathbf{J}}}).T_{jM}(m) &= T_{jM}(E_{\delta-\theta_{\mathbf{J}}}.m) \\ &= \sum_{a,b,c\geq 0;-a+b-c=n+\langle\alpha_j^\vee,\delta-\theta_{\mathbf{J}}\rangle} (-1)^b q_j^{b-ac} E_j^{(a)} F_j^{(b)} E_j^{(c)} E_{\delta-\theta_{\mathbf{J}}}.m \\ &= h_{\mathbf{J}}\left(\sum_{a,b,c\geq 0;-a+b-c=n+\langle\alpha_j^\vee,\delta-\theta_{\mathbf{J}}\rangle} (-1)^b q_j^{b-ac} \check{E}_j^{(a)} \check{F}_j^{(b)} \check{E}_j^{(c)} \check{E}_{\delta-\theta_{\mathbf{J}}}\right).m \\ &= h_{\mathbf{J}}\left(\check{T}_j(\check{E}_{\delta-\theta_{\mathbf{J}}}) \sum_{a,b,c\geq 0;-a+b-c=n} (-1)^b q_j^{b-ac} \check{E}_j^{(a)} \check{F}_j^{(b)} \check{E}_j^{(c)}\right).m \\ &= h_{\mathbf{J}}(\check{T}_j(\check{E}_{\delta-\theta_{\mathbf{J}}})).T_{jM}(m) = h_{\mathbf{J}}\circ\check{T}_j\circ h_{\mathbf{J}}^{-1}(E_{\delta-\theta_{\mathbf{J}}}).T_{jM}(m) \end{aligned}$$

for all $n\in\mathbb{Z}$ and $m\in M_j^n$. Thus we get $T_j(E_{\delta-\theta_{\mathbf{J}}}) = h_{\mathbf{J}}\circ\check{T}_j\circ h_{\mathbf{J}}^{-1}(E_{\delta-\theta_{\mathbf{J}}})$ by Proposition 3.5.4 of [14]. Similarly, we see that $T_j(u) = h_{\mathbf{J}}\circ\check{T}_j\circ h_{\mathbf{J}}^{-1}(u)$ for $u = E_\alpha, F_\alpha, K_\alpha$ with $\alpha\in\Pi_{\mathbf{J}}$. Hence (5.19) is valid since the both sides of (5.19) are automorphisms of $U_{\mathbf{J}}$. □

Proposition 5.10. *Let us use the notations introduced in Definition 3.5. Let* $\mathbf{J}$ *be an arbitrary connected subset of* $\overset{\circ}{\mathbf{I}}$. *Then the following equalities hold*:

$$T_{\widetilde{s_\alpha}}(E_\alpha) = -F_\alpha K_\alpha,\quad T_{\widetilde{s_\alpha}}(E_\beta) = (ad_q E_\alpha)^{(-\mathrm{A}_{\alpha\beta})}.E_\beta\quad (\alpha\neq\beta), \tag{5.20}$$

$$T_{\widetilde{s_\alpha}}(F_\alpha) = -K_\alpha^{-1}E_\alpha,\quad T_{\widetilde{s_\alpha}}(F_\beta) = \Omega(T_{\widetilde{s_\alpha}}(E_\beta))\quad (\alpha\neq\beta), \tag{5.21}$$

$$T_{\widetilde{s_\alpha}}(K_\beta) = K_{s_\alpha(\beta)} = K_\beta K_\alpha^{-\mathrm{A}_{\alpha\beta}}, \tag{5.22}$$

where $\alpha,\beta\in\Pi_{\mathbf{J}}$. *In particular, the restriction of* $T_{\widetilde{s_\alpha}}|_{U_{\mathbf{J}}}$ *is an automorphism of* $U_{\mathbf{J}}$. *If, in addition,* $\mathbf{J}'$ *is a connected subsets of* $\overset{\circ}{\mathbf{I}}$ *which is disjointed with* $\mathbf{J}$, *then the following equalities hold*:

$$\text{(i)}\ \ T_{\widetilde{s_\alpha}}(X) = X,\quad \text{(ii)}\ \ [T_{\widetilde{\tau}}, T_{\widetilde{s_{\alpha'}}}] = 0,\quad \text{(iii)}\ \ [T_{\widetilde{s_\alpha}}, T_{\widetilde{s_{\alpha'}}}] = 0. \tag{5.23}$$

for all $X\in U_{\mathbf{J}'}$, $(\alpha,\alpha')\in\Pi_{\mathbf{J}}\times\Pi_{\mathbf{J}'}$, *and* $\tau\in\Omega_{\mathbf{J}}$.

Proof. By Lemma 2.7(5) and Definition 3.5, we have

$$T_{\widetilde{s_{\delta-\theta_{\mathbf{J}}}}} = T_{(\varepsilon_{j_0})^{\mathbf{J}}} T_{j_0^-} T_{(\varepsilon_{j_0^-})^{\mathbf{J}}}, \tag{5.24}$$

where j_0 and j_0^- are the fixed elements of $\mathbf{J}_*$ such that $\delta-\theta_{\mathbf{J}} = \rho_{\mathbf{J}j_0}(\alpha_{j_0^-})$ and $(\rho_{\mathbf{J}j_0})^{-1} = \rho_{\mathbf{J}j_0^-}$.

Let us prove (5.22). In the case where $\alpha=\alpha_j$ with $j\in\mathbf{J}$, the equality is clear since $T_{\widetilde{s_\alpha}} = T_j$. In the case where $\alpha = \delta-\theta_{\mathbf{J}}$, (2.14) implies (5.22).

Let us prove the left equality of (5.20) and (5.21). In the case where $\alpha=\alpha_j$ with $j\in\mathbf{J}$, the equalities are clear since $T_{\widetilde{s_\alpha}} = T_j$. In the case where $\alpha=\delta-\theta_{\mathbf{J}}$, it follows from (5.2), (5.24), and Proposition 5.6(1) that

$$T_{\widetilde{s_\alpha}}(E_\alpha) = T_{(\varepsilon_{j_0})^{\mathbf{J}}} T_{j_0^-}(E_{j_0^-}) = T_{(\varepsilon_{j_0})^{\mathbf{J}}}(-F_{j_0^-}K_{j_0^-}) = -F_\alpha K_\alpha.$$

Since $\Omega T_{\widetilde{s_\alpha}} = T_{\widetilde{s_\alpha}}\Omega$, we have $T_{\widetilde{s_\alpha}}(F_\alpha) = -K_\alpha^{-1}E_\alpha$.

Let us prove the right equality of (5.20) and (5.21). Since $F_\beta = \Omega(E_\beta)$, the right equality of (5.21) follows from the right equality of (5.20) and the left equality of

(4.21). Hence it suffices to prove the right equality of (5.20). In the case where $\alpha = \alpha_j$ with $j \in \mathbf{J}$, since $T_{\widetilde{s_\alpha}} = T_j$, it follows from Lemma 5.9 that

$$\begin{aligned} T_{\widetilde{s_\alpha}}(E_\beta) &= T_j(E_\beta) = h_{\mathbf{J}} \circ \check{T}_j \circ h_{\mathbf{J}}^{-1}(E_\beta) = h_{\mathbf{J}} \circ \check{T}_j(\check{E}_\beta) \\ &= h_{\mathbf{J}}\left((ad_q \check{E}_j)^{(-\mathrm{A}_{\alpha_j\beta})}.\check{E}_\beta\right) = (ad_q E_j)^{(-\mathrm{A}_{\alpha_j\beta})}.E_\beta = (ad_q E_\alpha)^{(-\mathrm{A}_{\alpha\beta})}.E_\beta. \end{aligned}$$

In the case where $\alpha = \delta - \theta_{\mathbf{J}}$ and $\beta = \alpha_j$ with $j \in \mathbf{J}$, set $\gamma := \rho_{\mathbf{J}j_0}^{-1}(\beta) = \rho_{\mathbf{J}j_0}^{-1}(\alpha_j)$, then we see that $T_{(\varepsilon_{j_0^-})^{\mathbf{J}}}(E_\beta) = E_\gamma$ and $T_{(\varepsilon_{j_0})^{\mathbf{J}}}(E_\gamma) = E_\beta$, and hence

$$\begin{aligned} T_{\widetilde{s_\alpha}}(E_\beta) &= T_{(\varepsilon_{j_0})^{\mathbf{J}}} T_{j_0^-} T_{(\varepsilon_{j_0^-})^{\mathbf{J}}}(E_\beta) = T_{(\varepsilon_{j_0})^{\mathbf{J}}} T_{j_0^-}(E_\gamma) \\ &= T_{(\varepsilon_{j_0})^{\mathbf{J}}}\left((ad_q E_{j_0^-})^{(-\mathrm{A}_{j_0^-\gamma})}.E_\gamma\right) = (ad_q E_\alpha)^{(-\mathrm{A}_{\alpha\beta})}.E_\beta, \end{aligned}$$

where $\mathrm{A}_{j_0^-\gamma} = 2(\alpha_{j_0^-}|\gamma)/(\alpha_{j_0^-}|\alpha_{j_0^-})$.

The (i) of (5.23) follows from Lemma 5.4(3)(ii)(iii), and the (ii)(iii) of (5.23) follow from Lemma 5.4(2)(ii)(iii). □

Definition 5.11. For each non-empty subset $\mathbf{J} \subset \overset{\circ}{\mathbf{I}}$, we define $\mathcal{B}_{\widehat{W}_{\mathbf{J}}}$ to be the braid group associated with $\widehat{W}_{\mathbf{J}}$ by the generators $\{{}_{\mathbf{J}}T_{s_\alpha}, {}_{\mathbf{J}}T_\tau \,|\, \alpha \in \Pi_{\mathbf{J}}, \tau \in \Omega_{\mathbf{J}}\}$ and the following fundamental relations:

(i) ${}_{\mathbf{J}}T_{s_\alpha} \cdot {}_{\mathbf{J}}T_{s_\beta} = {}_{\mathbf{J}}T_{s_\beta} \cdot {}_{\mathbf{J}}T_{s_\alpha}$ if $\mathrm{ord}(s_\alpha s_\beta) = 2$,

(ii) ${}_{\mathbf{J}}T_{s_\alpha} \cdot {}_{\mathbf{J}}T_{s_\beta} \cdot {}_{\mathbf{J}}T_{s_\alpha} = {}_{\mathbf{J}}T_{s_\beta} \cdot {}_{\mathbf{J}}T_{s_\alpha} \cdot {}_{\mathbf{J}}T_{s_\beta}$ if $\mathrm{ord}(s_\alpha s_\beta) = 3$,

(iii) $({}_{\mathbf{J}}T_{s_\alpha} \cdot {}_{\mathbf{J}}T_{s_\beta})^2 = ({}_{\mathbf{J}}T_{s_\beta} \cdot {}_{\mathbf{J}}T_{s_\alpha})^2$ if $\mathrm{ord}(s_\alpha s_\beta) = 4$,

(iv) $({}_{\mathbf{J}}T_{s_\alpha} \cdot {}_{\mathbf{J}}T_{s_\beta})^3 = ({}_{\mathbf{J}}T_{s_\beta} \cdot {}_{\mathbf{J}}T_{s_\alpha})^3$ if $\mathrm{ord}(s_\alpha s_\beta) = 6$,

(v) ${}_{\mathbf{J}}T_\tau \cdot {}_{\mathbf{J}}T_{s_\alpha} = {}_{\mathbf{J}}T_{s_{\tau(\alpha)}} \cdot {}_{\mathbf{J}}T_\tau$, (vi) ${}_{\mathbf{J}}T_\tau \cdot {}_{\mathbf{J}}T_{\tau'} = {}_{\mathbf{J}}T_{\tau\tau'}$, (vii) ${}_{\mathbf{J}}T_1 = 1$,

where $\mathrm{ord}(x)$ is the order of x. The braid group $\mathcal{B}_{\widehat{W}_{\mathbf{J}}}$ is also defined by the generators $\{{}_{\mathbf{J}}T_x \,|\, x \in \widehat{W}_{\mathbf{J}}\}$ and the following fundamental relations:

$$ {}_{\mathbf{J}}T_x \cdot {}_{\mathbf{J}}T_y = {}_{\mathbf{J}}T_{xy} \quad \text{if } \ell_{\mathbf{J}}(x) + \ell_{\mathbf{J}}(y) = \ell_{\mathbf{J}}(xy), $$

where $\ell_{\mathbf{J}} \colon \widehat{W}_{\mathbf{J}} \to \mathbb{Z}_+$ is the extended length function. In the case where $\mathbf{J} = \overset{\circ}{\mathbf{I}}$, we can denote ${}_{\mathbf{J}}T_x$ simply by T_x.

Theorem 5.12. *Let us use the notations introduced in Definition 3.5. For each non-empty subset $\mathbf{J} \subset \overset{\circ}{\mathbf{I}}$, the braid group $\mathcal{B}_{\widehat{W}_{\mathbf{J}}}$ acts on $U_{\mathbf{J}}$ as a group of $\mathbb{Q}(q)$-algebra automorphisms of $U_{\mathbf{J}}$ via*

$$ {}_{\mathbf{J}}T_s \mapsto T_{\widetilde{s}}|_{U_{\mathbf{J}}}, \tag{5.25} $$

where $s \in \widehat{S}_{\mathbf{J}}$. Moreover, the action of ${}_{\mathbf{J}}T_x$ on $U_{\mathbf{J}}$ is given by

$$ {}_{\mathbf{J}}T_x \mapsto T_{[\widetilde{\boldsymbol{s}}]}|_{U_{\mathbf{J}}} \tag{5.26} $$

for each $x \in \widehat{W}_{\mathbf{J}}$, where $\boldsymbol{s}$ is an element of $\widehat{\mathcal{W}}_{\mathbf{J}}$ such that $[\boldsymbol{s}] = x$.

Proof. By direct calculations as in the section 39.2 of the Lusztig's book [14] using Proposition 5.5–5.7 and 5.10, we see that the automorphisms $T_{\widetilde{s}}|_{U_{\mathbf{J}}}$ satisfy the fundamental relations (i)–(vii) of $\mathcal{B}_{\widehat{W}_{\mathbf{J}}}$ with replacing ${}_{\mathbf{J}}T_s$ by $T_{\widetilde{s}}|_{U_{\mathbf{J}}}$, and hence the assignment (5.25) defines a group homomorphism from $\mathcal{B}_{\widehat{W}_{\mathbf{J}}}$ to the automorphism group $\mathrm{Aut}(U_{\mathbf{J}})$.

We next prove (5.26). Denote the $\boldsymbol{s}$ by $\boldsymbol{s} = (\boldsymbol{s}(p))_{p\in\mathbb{N}_n}$ with $n \in \mathbb{N}$. Then

$$x = [\boldsymbol{s}] = \boldsymbol{s}(1)\boldsymbol{s}(2)\cdots\boldsymbol{s}(n), \quad \ell_{\mathbf{J}}(x) = \ell_{\mathbf{J}}(\boldsymbol{s}(1)) + \ell_{\mathbf{J}}(\boldsymbol{s}(2)) + \cdots + \ell_{\mathbf{J}}(\boldsymbol{s}(n)),$$

and hence the following equality in $\mathcal{B}_{\widehat{W}_{\mathbf{J}}}$ holds:

$${}_{\mathbf{J}}T_x = {}_{\mathbf{J}}T_{\boldsymbol{s}(1)} \cdot {}_{\mathbf{J}}T_{\boldsymbol{s}(2)} \cdots {}_{\mathbf{J}}T_{\boldsymbol{s}(n)}.$$

By (3.6)(3.11), we see that

$$[\widetilde{\boldsymbol{s}}] = \widetilde{\boldsymbol{s}(1)}\widetilde{\boldsymbol{s}(2)}\cdots\widetilde{\boldsymbol{s}(n)}, \quad \ell([\widetilde{\boldsymbol{s}}]) = \ell(\widetilde{\boldsymbol{s}(1)}) + \ell(\widetilde{\boldsymbol{s}(2)}) + \cdots + \ell(\widetilde{\boldsymbol{s}(n)}),$$

which imply the following equality in $\mathcal{B}_{\widehat{W}}$:

$$T_{[\widetilde{\boldsymbol{s}}]} = T_{\widetilde{\boldsymbol{s}(1)}} \cdot T_{\widetilde{\boldsymbol{s}(2)}} \cdots T_{\widetilde{\boldsymbol{s}(n)}}.$$

Thus we get the following equality:

$$T_{[\widetilde{\boldsymbol{s}}]}|_{U_{\mathbf{J}}} = T_{\widetilde{\boldsymbol{s}(1)}}|_{U_{\mathbf{J}}} \cdot T_{\widetilde{\boldsymbol{s}(2)}}|_{U_{\mathbf{J}}} \cdots T_{\widetilde{\boldsymbol{s}(n)}}|_{U_{\mathbf{J}}}.$$

Therefore the action of ${}_{\mathbf{J}}T_x$ on $U_{\mathbf{J}}$ is given by (5.26). □

Remark 5.13. *Note that* ${}_{\mathbf{J}}T_w(u) = T_w(u)$ *for each* $w \in \mathring{W}_{\mathbf{J}}$ *and* $u \in U_{\mathbf{J}}$ *and that if* $\mathbf{J} = \mathring{\mathbf{I}}$ *then* ${}_{\mathbf{J}}T_x(u) = T_x(u)$ *for each* $x \in \widehat{W}_{\mathbf{J}}$ *and* $u \in U_{\mathbf{J}}$. *In Proposition 5.20, we will prove that the action of* $\mathcal{B}_{\widehat{W}_{\mathbf{J}}}$ *on* $U_{\mathbf{J}}$ *is faithful.*

Lemma 5.14. *Let* $\mathbf{J}$ *be an arbitrary non-empty subset of* $\mathring{\mathbf{I}}$, *and* $\mathbf{K}$ *an arbitrary connected subset of* $\mathbf{J}$.

(1) *The equality* $[{}_{\mathbf{J}}T_x, T_{\varepsilon_i}] = 0$ *in* $\mathrm{Aut}(U_{\mathbf{J}})$ *holds for all* $x \in \widehat{W}_{\mathbf{J}}$ *and* $i \in \mathring{\mathbf{I}} \setminus \mathbf{J}$.

(2) *For each* $k \in \mathbf{K}_*$, *we have*

$$E_{\delta-\theta_{\mathbf{K}}} = {}_{\mathbf{J}}T_{\varepsilon_k}T^{-1}_{(\varepsilon_k)_{\mathbf{K}}}(E_{k^-}), \tag{5.27}$$

where k^- *is a unique element of* $\mathbf{K}_*$ *such that* $\rho_{\mathbf{K}k}(\alpha_{k^-}) = \delta - \theta_{\mathbf{K}}$. *In particular, we have* $E_{\delta-\theta_{\mathbf{K}}} \in U^+_{\mathbf{J}}$ *and the following inclusions*:

$$U_{\mathbf{K}} \subset U_{\mathbf{J}}, \qquad U^+_{\mathbf{K}} \subset U^+_{\mathbf{J}}, \qquad U^-_{\mathbf{K}} \subset U^-_{\mathbf{J}}, \qquad U^0_{\mathbf{K}} \subset U^0_{\mathbf{J}}. \tag{5.28}$$

Proof. (1) This follows from Lemma 5.4(2)(i) and the equality $[T_j, T_{\varepsilon_i}] = 0$ in $\mathrm{Aut}(U_{\mathbf{J}})$ for all $j \in \mathbf{J}$.

(2) Let $\boldsymbol{s}$ be an element of $\widehat{\mathcal{W}}_{\mathbf{J}}$ such that $[\boldsymbol{s}] = t_{\varepsilon_k}$. Then, by Theorem 5.12, we have the following equality in $\mathrm{Aut}(U_{\mathbf{J}})$:

$${}_{\mathbf{J}}T_{\varepsilon_k} = T_{[\widetilde{\boldsymbol{s}}]}|_{U_{\mathbf{J}}}. \tag{5.29}$$

By (2.5)(i), (2.13)(i) in Lemma 2.7, and (3.20)(i) in Lemma 3.9, we have

$$[\widetilde{\boldsymbol{s}}]^{\mathbf{K}}(\alpha_{k^-}) = \delta - \theta_{\mathbf{K}} = (\varepsilon_k)^{\mathbf{K}}(\alpha_{k^-}).$$

Since $\Phi(\varepsilon_k) \subset \Delta(1,-)$, we have $\Phi((\varepsilon_k)^{\mathbf{K}}) \subset \Delta(1,-)$. Moreover, by (3.12)(3.13), we have $\Phi([\widetilde{\boldsymbol{s}}]) \subset \Delta(1,-)$, hence $\Phi([\widetilde{\boldsymbol{s}}]^{\mathbf{K}}) \subset \Delta(1,-)$. Therefore, by Lemma 5.1 and Definition 5.2, we have

$$T_{[\widetilde{\boldsymbol{s}}]^{\mathbf{K}}}(E_{k^-}) = E_{\delta-\theta_{\mathbf{K}}} = T_{(\varepsilon_k)^{\mathbf{K}}}(E_{k^-}). \tag{5.30}$$

By Lemma 3.9(3.20)(ii), we have

$$T_{[\widetilde{\boldsymbol{s}}]} = T_{[\widetilde{\boldsymbol{s}}]^{\mathbf{K}}}T_{(\varepsilon_k)_{\mathbf{K}}}. \tag{5.31}$$

Since $T^{-1}_{(\varepsilon_k)_{\mathbf{K}}}(E_{k^-})$, by (5.29)(5.30)(5.31), we get

$$E_{\delta-\theta_{\mathbf{K}}} = T_{[\widetilde{\boldsymbol{s}}]}|_{U_{\mathbf{J}}}T^{-1}_{(\varepsilon_k)_{\mathbf{K}}}(E_{k^-}) = {}_{\mathbf{J}}T_{\varepsilon_k}T^{-1}_{(\varepsilon_k)_{\mathbf{K}}}(E_{k^-}).$$

□

Proposition 5.15. *For each non-empty subset* $\mathbf{J} \subset \overset{\circ}{\mathbf{I}}$ *and* $j \in \mathbf{J}$, *we have*

$$ {}_{\mathbf{J}}T_{\varepsilon_j} = T_{\varepsilon_j}|_{U_{\mathbf{J}}}. \tag{5.32}$$

Proof. Put $\mathbf{K} = \{j\}$. Then we see that $\theta_{\mathbf{K}} = \alpha_j$, $(\varepsilon_j)_{\mathbf{K}} = s_j$, and $j^- = j$. By Lemma 5.14(2), we have

$$E_{\delta-\alpha_j} = {}_{\mathbf{J}}T_{\varepsilon_j} T_j^{-1}(E_j). \tag{5.33}$$

Suppose that $j \in \mathbf{J}_*$. Then we see that

$$\varepsilon_j = \rho_{\mathbf{J}j}(\varepsilon_j)_{\mathbf{J}}, \quad \ell_{\mathbf{J}}(\varepsilon_j) = \ell_{\mathbf{J}}(\rho_{\mathbf{J}j}) + \ell_{\mathbf{J}}((\varepsilon_j)_{\mathbf{J}}), \quad \widetilde{\rho_{\mathbf{J}j}} = (\varepsilon_j)^{\mathbf{J}}.$$

Since ${}_{\mathbf{J}}T_{\varepsilon_j} = {}_{\mathbf{J}}T_{\rho_{\mathbf{J}j}} \cdot {}_{\mathbf{J}}T_{(\varepsilon_j)_{\mathbf{J}}}$, we have

$${}_{\mathbf{J}}T_{\varepsilon_j} = T_{(\varepsilon_j)^{\mathbf{J}}}|_{U_{\mathbf{J}}} \cdot T_{(\varepsilon_j)_{\mathbf{J}}}|_{U_{\mathbf{J}}} = T_{\varepsilon_j}|_{U_{\mathbf{J}}}.$$

We next suppose that $j \in \mathbf{J} \setminus \mathbf{J}_*$. It suffices to show that

$${}_{\mathbf{J}}T_{\varepsilon_j}(X) = T_{\varepsilon_j}(X) \tag{5.34}$$

for $X = E_\alpha, F_\alpha, K_\alpha^{\pm 1}$ with $\alpha \in \Pi_{\mathbf{J}}$. In the case where $X = K_\alpha^{\pm 1}$, the (5.34) is clear. By (5.33), we have

$${}_{\mathbf{J}}T_{\varepsilon_j}(F_j) = -K_{\delta-\alpha_j}^{-1} E_{\delta-\alpha_j} = T_{\varepsilon_j}(F_j)$$

for each $j \in \mathbf{J}$. In the case where $j' \in \mathbf{J} \setminus \{j\}$, we have $t_{\varepsilon_j}(\alpha_{j'}) = \alpha_{j'}$, and hence

$${}_{\mathbf{J}}T_{\varepsilon_j}(F_{j'}) = F_{j'} = T_{\varepsilon_j}(F_{j'}).$$

Thus (5.34) holds in the case where $X = F_\alpha$ with $\alpha \in \overset{\circ}{\Pi}_{\mathbf{J}}$. Since $[\Omega, {}_{\mathbf{J}}T_{\varepsilon_j}] = 0$, the (5.34) holds in the case where $X = E_\alpha$ with $\alpha \in \overset{\circ}{\Pi}_{\mathbf{J}}$. Therefore we have

$${}_{\mathbf{J}}T_{\varepsilon_j}|_{\overset{\circ}{U}_{\mathbf{J}}} = T_{\varepsilon_j}|_{\overset{\circ}{U}_{\mathbf{J}}},$$

where $\overset{\circ}{U}_{\mathbf{J}}$ is the $\mathbb{Q}(q)$-subalgebra of $U_{\mathbf{J}}$ generated by $\{E_\alpha, F_\alpha, K_\alpha^{\pm 1} \,|\, \alpha \in \overset{\circ}{\Pi}_{\mathbf{J}}\}$. We next prove (5.34) in the case where $X = E_{\delta-\theta_{\mathbf{J}}}$. By Lemma 5.14(2), we have

$$E_{\delta-\theta_{\mathbf{J}}} = {}_{\mathbf{J}}T_{\varepsilon_k} T_{(\varepsilon_k)_{\mathbf{J}}}^{-1}(E_{k^-}),$$

where $k, k^- \in \mathbf{J}_*$ such that $\rho_{\mathbf{J}k}(\alpha_{k^-}) = \delta - \theta_{\mathbf{J}}$. Since $k \in \mathbf{J}_*$, we have

$${}_{\mathbf{J}}T_{\varepsilon_k} = T_{\varepsilon_k}|_{U_{\mathbf{J}}}.$$

Since $T_{(\varepsilon_k)_{\mathbf{J}}}^{-1}(E_{k^-}) \in \overset{\circ}{U}_{\mathbf{J}}$, we have

$${}_{\mathbf{J}}T_{\varepsilon_j} T_{(\varepsilon_k)_{\mathbf{J}}}^{-1}(E_{k^-}) = T_{\varepsilon_j} T_{(\varepsilon_k)_{\mathbf{J}}}^{-1}(E_{k^-}).$$

In addition, it is clear that $[{}_{\mathbf{J}}T_{\varepsilon_j}\,, {}_{\mathbf{J}}T_{\varepsilon_k}] = 0$. Therefore we see that

$$\begin{aligned} {}_{\mathbf{J}}T_{\varepsilon_j}(E_{\delta-\theta_{\mathbf{J}}}) &= {}_{\mathbf{J}}T_{\varepsilon_k} \cdot {}_{\mathbf{J}}T_{\varepsilon_j} T_{(\varepsilon_k)_{\mathbf{J}}}^{-1}(E_{k^-}) = {}_{\mathbf{J}}T_{\varepsilon_k} T_{\varepsilon_j} T_{(\varepsilon_k)_{\mathbf{J}}}^{-1}(E_{k^-}) \\ &= T_{\varepsilon_k} T_{\varepsilon_j} T_{(\varepsilon_k)_{\mathbf{J}}}^{-1}(E_{k^-}) = T_{\varepsilon_j} T_{\varepsilon_k} T_{(\varepsilon_k)_{\mathbf{J}}}^{-1}(E_{k^-}) = T_{\varepsilon_j}(E_{\delta-\theta_{\mathbf{J}}}). \end{aligned}$$

The (5.34) for $X = F_{\delta-\theta_{\mathbf{J}}}$ also holds, since $[\Omega, {}_{\mathbf{J}}T_{\varepsilon_j}] = 0$. □

Definition 5.16. Let $\mathbf{J}$ be an arbitrary non-empty subset of $\mathring{\mathbf{I}}$. For each $y \in \widehat{W}_{\mathbf{J}}$, we define $\mathbb{Q}(q)$-subalgebras $A_{\mathbf{J}}(y)$ and $A_{\mathbf{J}}(y)^c$ of $U_{\mathbf{J}}^+$ by setting

$$A_{\mathbf{J}}(y) := \{\, u \in U_{\mathbf{J}}^+ \mid {}_{\mathbf{J}}T_y^{-1}(u) \in U_{\mathbf{J}}^{\leq 0} \,\}, \quad A_{\mathbf{J}}(y)^c := \{\, u \in U_{\mathbf{J}}^+ \mid {}_{\mathbf{J}}T_y^{-1}(u) \in U_{\mathbf{J}}^+ \,\}. \tag{5.35}$$

Note that $A_{\mathbf{J}}(y) = A_{\mathbf{J}}(|y|)$ and $A_{\mathbf{J}}(y)^c = A_{\mathbf{J}}(|y|)^c$, where $y = |y|\tau_y$, $|y| \in W_{\mathbf{J}}$, and $\tau_y \in \Omega_{\mathbf{J}}$. For each $B \in \mathfrak{B}_{\mathbf{J}}^*$, we set

$$A_{\mathbf{J}}(B) := \bigcup_{y \in W_{\mathbf{J}}(B)} A_{\mathbf{J}}(y), \quad A_{\mathbf{J}}(B)^c := \bigcap_{y \in W_{\mathbf{J}}(B)} A_{\mathbf{J}}(y)^c, \tag{5.36}$$

where $W_{\mathbf{J}}(B) = \{\, y \in W_{\mathbf{J}} \,|\, \Phi_{\mathbf{J}}(y) \subset B \,\}$. Here note that $A_{\mathbf{J}}(B)^c$ is a $\mathbb{Q}(q)$-subalgebra of $U_{\mathbf{J}}^+$. In Proposition 7.2(2), we will show that $A_{\mathbf{J}}(B)$ is also a $\mathbb{Q}(q)$-subalgebra of $U_{\mathbf{J}}^+$. For each $w \in \mathring{W}_{\mathbf{J}}$, we set

$$A_{\mathbf{J}}(w,-) := A_{\mathbf{J}}(\Delta_{\mathbf{J}}(w,-)), \quad A_{\mathbf{J}}(w,-)^c := A_{\mathbf{J}}(\Delta_{\mathbf{J}}(w,-))^c, \tag{5.37}$$

$$A_{\mathbf{J}}(w,+) := \Psi A_{\mathbf{J}}(\Delta_{\mathbf{J}}(w,+)), \quad A_{\mathbf{J}}(w,+)^c := \Psi A_{\mathbf{J}}(\Delta_{\mathbf{J}}(w,+))^c. \tag{5.38}$$

Note that

$$A_{\mathbf{J}}(w,-) = \{\, u \in U_{\mathbf{J}}^+ \mid \exists y \in W_{\mathbf{J}}(\Delta_{\mathbf{J}}(w,-));\ {}_{\mathbf{J}}T_y^{-1}(u) \in U_{\mathbf{J}}^{\leq 0} \,\}, \tag{5.39}$$

$$A_{\mathbf{J}}(w,-)^c = \{\, u \in U_{\mathbf{J}}^+ \mid \forall y \in W_{\mathbf{J}}(\Delta_{\mathbf{J}}(w,-)),\ {}_{\mathbf{J}}T_y^{-1}(u) \in U_{\mathbf{J}}^+ \,\}, \tag{5.40}$$

$$A_{\mathbf{J}}(w,+) = \{\, u \in U_{\mathbf{J}}^+ \mid \exists y \in W_{\mathbf{J}}(\Delta_{\mathbf{J}}(w,+));\ {}_{\mathbf{J}}T_{y^{-1}}(u) \in U_{\mathbf{J}}^{\leq 0} \,\}, \tag{5.41}$$

$$A_{\mathbf{J}}(w,+)^c = \{\, u \in U_{\mathbf{J}}^+ \mid \forall y \in W_{\mathbf{J}}(\Delta_{\mathbf{J}}(w,+)),\ {}_{\mathbf{J}}T_{y^{-1}}(u) \in U_{\mathbf{J}}^+ \,\}. \tag{5.42}$$

In addition, we define a $\mathbb{Q}(q)$-subalgebra $A_{\mathbf{J}}(w,0)$ of $U_{\mathbf{J}}^+$ by setting

$$A_{\mathbf{J}}(w,0) := A_{\mathbf{J}}(w,-)^c \cap A_{\mathbf{J}}(w,+)^c. \tag{5.43}$$

In the case where $\mathbf{J} = \mathring{\mathbf{I}}$, we will denote the symbols above more simply by removing $\mathbf{J}$ from them.

Lemma 5.17. (1) *For each $y \in W_{\mathbf{J}}$, the multiplication defines the following injective $\mathbb{Q}(q)$-linear mappings*:

$$A_{\mathbf{J}}(y) \otimes A_{\mathbf{J}}(y)^c \hookrightarrow U_{\mathbf{J}}^+, \quad A_{\mathbf{J}}(y)^c \otimes A_{\mathbf{J}}(y) \hookrightarrow U_{\mathbf{J}}^+. \tag{5.44}$$

(2) *Let $w_\circ$ be the longest element of $\mathring{W}_{\mathbf{J}}$. Then the following equalities hold*:

$$A_{\mathbf{J}}(w,+) = \Psi A_{\mathbf{J}}(ww_\circ,-), \quad A_{\mathbf{J}}(w,+)^c = \Psi A_{\mathbf{J}}(ww_\circ,-)^c, \tag{5.45}$$

$$A_{\mathbf{J}}(w,0) = \Psi A_{\mathbf{J}}(ww_\circ,0). \tag{5.46}$$

Proof. (1) Since ${}_{\mathbf{J}}T_y$ is an automorphism of the $\mathbb{Q}(q)$-algebra $U_{\mathbf{J}}$, the assignment $a \otimes b \mapsto {}_{\mathbf{J}}T_y^{-1}(a) \otimes {}_{\mathbf{J}}T_y^{-1}(b)$ defines an automorphism $({}_{\mathbf{J}}T_y^{-1})^{\otimes 2}$ of the $\mathbb{Q}(q)$-algebra $U_{\mathbf{J}}^{\otimes 2}$. Let m be the multiplication mapping $A_{\mathbf{J}}(y) \otimes A_{\mathbf{J}}(y)^c \to U_{\mathbf{J}}^+$, and m' the multiplication mapping $U_{\mathbf{J}}^{\leq 0} \otimes U_{\mathbf{J}}^+ \simeq U_{\mathbf{J}}$. Then we see that

$${}_{\mathbf{J}}T_y^{-1} \circ m = m' \circ ({}_{\mathbf{J}}T_y^{-1})^{\otimes 2}|_{A_{\mathbf{J}}(y) \otimes A_{\mathbf{J}}(y)^c},$$

where $({}_{\mathbf{J}}T_y^{-1})^{\otimes 2}|_{A_{\mathbf{J}}(y) \otimes A_{\mathbf{J}}(y)^c}$ is the restriction of $({}_{\mathbf{J}}T_y^{-1})^{\otimes 2}$ to $A_{\mathbf{J}}(y) \otimes A_{\mathbf{J}}(y)^c$. Since the right hand side is injective, we see that m is injective. The proof of the remains is quite similar.

(2) The equality $\Delta_{\mathbf{J}}(w,+) = \Delta_{\mathbf{J}}(ww_\circ,-)$ implies (5.45). From (5.45), it follows that

$$\Psi A_{\mathbf{J}}(ww_\circ,+)^c = \Psi\Psi A_{\mathbf{J}}(ww_\circ w_\circ,-)^c = A_{\mathbf{J}}(w,-)^c.$$

Hence, by (5.43), we see that

$$\Psi A_{\mathbf{J}}(ww_\circ,0) = \Psi A_{\mathbf{J}}(ww_\circ,-)^c \cap \Psi A_{\mathbf{J}}(ww_\circ,+)^c = A_{\mathbf{J}}(w,+)^c \cap A_{\mathbf{J}}(w,-)^c = A_{\mathbf{J}}(w,0).$$

□

Definition 5.18. For each $s \in S_{\mathbf{J}}$, we set

$$E_s := \begin{cases} E_{\delta-\theta_{\mathbf{J}_c}} & \text{if } s = s_{\delta-\theta_{\mathbf{J}_c}} \text{ with } c = 1,\dots,\mathrm{C}(\mathbf{J}), \\ E_j & \text{if } s = s_j \text{ with } j \in \mathbf{J}, \end{cases} \tag{5.47}$$

where $E_{\delta-\theta_{\mathbf{J}_c}}$ is introduced in Definition 5.2. For each $\boldsymbol{s} \in \mathcal{W}_{\mathbf{J}}^*$ and $p \in \mathbb{N}_{\ell(\boldsymbol{s})}$, we define a weight vector $E_{\boldsymbol{s}}(p)$ of $U_{\mathbf{J}}^+$ with weight $\phi_{\boldsymbol{s}}(p)$ by setting

$$E_{\boldsymbol{s}}(p) := {}_{\mathbf{J}}T_{\boldsymbol{s}(1)} \cdot \dots \cdot {}_{\mathbf{J}}T_{\boldsymbol{s}(p-1)}(E_{\boldsymbol{s}(p)}). \tag{5.48}$$

If $\phi_{\boldsymbol{s}}(p) = \beta$, we denote the $E_{\boldsymbol{s}}(p)$ by $E_{\beta,\boldsymbol{s}}$.

For each function $f\colon X \to \mathbb{Z}_+$, we set $\mathrm{supp}(f) := \{x \in X \mid f(x) > 0\}$, and call the set $\mathrm{supp}(f)$ the *support* of f. If $\sharp\mathrm{supp}(f) < \infty$, we call f a *finitely supported function*.

For each finitely supported function $\mathbf{c}\colon \mathbb{N}_{\ell(\boldsymbol{s})} \to \mathbb{Z}_+$, we set

$$E_{\boldsymbol{s},<}^{\mathbf{c}} := E_{\boldsymbol{s}}(p_1)^{\mathbf{c}(p_1)} \cdot E_{\boldsymbol{s}}(p_2)^{\mathbf{c}(p_2)} \cdot \dots \cdot E_{\boldsymbol{s}}(p_m)^{\mathbf{c}(p_m)}, \tag{5.49}$$

$$E_{\boldsymbol{s},>}^{\mathbf{c}} := E_{\boldsymbol{s}}(p_m)^{\mathbf{c}(p_m)} \cdot \dots \cdot E_{\boldsymbol{s}}(p_2)^{\mathbf{c}(p_2)} \cdot E_{\boldsymbol{s}}(p_1)^{\mathbf{c}(p_1)}, \tag{5.50}$$

where $\mathrm{supp}(\mathbf{c}) = \{p_1, p_2, \dots, p_m\}$ with $p_1 < p_2 < \dots < p_m$. Here we set $E_{\boldsymbol{s},<}^{\mathbf{c}} = E_{\boldsymbol{s},>}^{\mathbf{c}} := 1$ in the case where $\mathrm{supp}(\mathbf{c}) = \emptyset$. Let us denote by $\mathcal{E}_{\boldsymbol{s},<}$ (resp. $\mathcal{E}_{\boldsymbol{s},>}$) the set of all $E_{\boldsymbol{s},<}^{\mathbf{c}}$ (resp. $E_{\boldsymbol{s},>}^{\mathbf{c}}$).

Proposition 5.19. (1) *Let B be a real biconvex set in $\Delta_{\mathbf{J}+}$, and $\boldsymbol{s}$ an element of $\mathcal{W}_{\mathbf{J}}^*$ such that $B = \Phi_{\mathbf{J}}^*([\boldsymbol{s}])$. Then the set $\mathcal{E}_{\boldsymbol{s},<}$ (resp. $\mathcal{E}_{\boldsymbol{s},>}$) form a basis of a subspace $U_{\mathbf{J},<}(B)$ (resp. $U_{\mathbf{J},>}(B)$) of $U_{\mathbf{J}}^+$ which does not depend on the choice of $\boldsymbol{s}$. Moreover, the multiplication defines the following injective $\mathbb{Q}(q)$-linear mappings*:

$$U_{\mathbf{J},<}(B) \otimes A_{\mathbf{J}}(B)^c \hookrightarrow U_{\mathbf{J}}^+, \quad U_{\mathbf{J},>}(B) \otimes A_{\mathbf{J}}(B)^c \hookrightarrow U_{\mathbf{J}}^+, \tag{5.51}$$

$$A_{\mathbf{J}}(B)^c \otimes U_{\mathbf{J},<}(B) \hookrightarrow U_{\mathbf{J}}^+, \quad A_{\mathbf{J}}(B)^c \otimes U_{\mathbf{J},>}(B) \hookrightarrow U_{\mathbf{J}}^+. \tag{5.52}$$

(2) *Let $\mathbf{J}_1$, $\mathbf{J}_2$ be non-empty subsets of $\mathring{\mathbf{I}}$ which are disjointed with each other, and (B_1, B_2) an element of $\mathfrak{B}_{\mathbf{J}_1}^* \times \mathfrak{B}_{\mathbf{J}_2}^*$. Then the multiplication defines the following injective $\mathbb{Q}(q)$-linear mappings*:

$$U_{\mathbf{J}_1,<}(B_1) \otimes U_{\mathbf{J}_2,<}(B_2) \hookrightarrow U^+, \quad U_{\mathbf{J}_1,>}(B_1) \otimes U_{\mathbf{J}_2,>}(B_2) \hookrightarrow U^+. \tag{5.53}$$

Proof. (1) We first consider the linear independence over $\mathbb{Q}(q)$ of the sets $\mathcal{E}_{\boldsymbol{s},<}$. Since the proof of the linear independence is similar to that of Lemma 8.21 in [10], so we omit the detailed proof, but we give a key point. Since E_j^k is a non-zero element of $A_{\mathbf{J}}(s_j)$ with weight $k\alpha_j$ for each $j \in \mathbf{J}$ and $k = 0, 1, \dots, m$, the elements E_j^k $(k = 0, 1, \dots, m)$ of $A_{\mathbf{J}}(s_j)$ are linearly independent over $\mathbb{Q}(q)$. Thus it follows from Lemma 5.17(1) that the equalities $\sum_{k=0}^m E_j^k u_k = 0$ with $u_k \in A_{\mathbf{J}}(s_j)^c$ $(k = 0, 1, \dots, m)$ imply that $u_k = 0$ for all k.

We next prove the independence of $U_{\mathbf{J},<}(B)$ from the choice of $\boldsymbol{s}$. For convenience, we denote by $U_{\mathbf{J},<}(\boldsymbol{s})$ the $\mathbb{Q}(q)$-subspace of $U_{\mathbf{J}}^+$ spanned by $\mathcal{E}_{\boldsymbol{s},<}$. Then it suffices to show that $U_{\mathbf{J},<}(\boldsymbol{s}) = U_{\mathbf{J},<}(\boldsymbol{s}')$ for another element $\boldsymbol{s}' \in \mathcal{W}^\infty$ such that $\Phi^\infty([\boldsymbol{s}']) = B$. In the case where B is a finite biconvex set in $\Delta_{\mathbf{J}+}$, since $\boldsymbol{s}$ is a finite reduced word, the proof of the assertion is similar to that of Proposition 8.22 in [10], so we omit that. We will prove the case where B is a infinite real biconvex set in $\Delta_{\mathbf{J}+}$. Since $\Phi^\infty_{\mathbf{J}}([\boldsymbol{s}]) = \Phi^\infty_{\mathbf{J}}([\boldsymbol{s}'])$, for each $(m,n) \in \mathbb{N}^2$, there exists $(m',n') \in \mathbb{Z}_{>m} \times \mathbb{Z}_{>n}$ such that $\Phi_{\mathbf{J}}([\boldsymbol{s}_{|m}]) \subset \Phi_{\mathbf{J}}([\boldsymbol{s}'_{|m'}])$and $\Phi_{\mathbf{J}}([\boldsymbol{s}'_{|n}]) \subset \Phi_{\mathbf{J}}([\boldsymbol{s}_{|n'}])$, which implies $U_{\mathbf{J},<}([\boldsymbol{s}_{|m}]) \subset U_{\mathbf{J},<}([\boldsymbol{s}'_{|m'}])$ and $U_{\mathbf{J},<}([\boldsymbol{s}_{|n}]) \subset U_{\mathbf{J},<}([\boldsymbol{s}'_{|n'}])$. Since $U_{\mathbf{J},<}(\boldsymbol{s}) = \cup_{p\in\mathbb{N}} U_{\mathbf{J},<}([\boldsymbol{s}_{|p}])$ and $U_{\mathbf{J},<}(\boldsymbol{s}') = \cup_{p\in\mathbb{N}} U_{\mathbf{J},<}([\boldsymbol{s}'_{|p}])$, we get $U_{\mathbf{J},<}(\boldsymbol{s}) = U_{\mathbf{J},<}(\boldsymbol{s}')$. Since the proof of the assertion for $\mathcal{E}_{\boldsymbol{S},>}$ is quite similar, we omit that.

We next prove (5.51) and (5.52). By Lemma 5.17(1), we see that the multiplication $A_{\mathbf{J}}([\boldsymbol{s}_{|p}]) \otimes A_{\mathbf{J}}([\boldsymbol{s}_{|p}])^c \to U^+_{\mathbf{J}}$ is injective for each $p \in \mathbb{N}_{\ell(\boldsymbol{S})}$. It is clear that $U_{\mathbf{J},<}(\boldsymbol{s}_{|p}) \subset A_{\mathbf{J}}([\boldsymbol{s}_{|p}])^c$ and $A_{\mathbf{J}}(B)^c \subset A_{\mathbf{J}}([\boldsymbol{s}_{|p}])^c$. It follows that the multiplication $m_p\colon U_{\mathbf{J},<}(\boldsymbol{s}_{|p}) \otimes A_{\mathbf{J}}(B)^c \to U^+_{\mathbf{J}}$ is injective. Suppose that two elements (a_1,b_1) and (a_1,b_2) of $U_{\mathbf{J},<}(\boldsymbol{s}) \times A_{\mathbf{J}}(B)^c$ satisfy $a_1b_1 = a_2b_2$. We may assume that both a_1 and a_2 belong to $U_{\mathbf{J},<}(\boldsymbol{s}_{|p})$ for some $p \in \mathbb{N}_{\ell(\boldsymbol{S})}$. Then the injectivity of m_p implies that $a_1 = a_2$ and $b_1 = b_2$. Therefore the multiplication mapping $U_{\mathbf{J},<}(B) \otimes A_{\mathbf{J}}(B)^c \to U^+_{\mathbf{J}}$ is injective. The proof of the remains are quite similar.

(2) Set $\mathbf{J} = \mathbf{J}_1 \amalg \mathbf{J}_2$. Then we see that B_1 is a real biconvex subset in $\Delta_{\mathbf{J}+}$. From (5.23)(i) in Proposition 5.10, it follows that both $U_{\mathbf{J}_2,<}(B_2)$ and $U_{\mathbf{J}_2,>}(B_2)$ are subspaces of $A_{\mathbf{J}}(B)^c$. Thus (5.53) follows from (5.51). □

Proposition 5.20. (1) *Suppose that B is a real biconvex set in $\Delta_{\mathbf{J}+}$ and set $B_c := B \cap \Delta_{\mathbf{J}_c}$ for each $c = 1, \ldots, \mathrm{C}(\mathbf{J})$. Then the multiplication define the following isomorphisms of $\mathbb{Q}(q)$-vector spaces*:

$$\otimes_{c=1}^{\mathrm{C}(\mathbf{J})} U_{\mathbf{J}_c,<}(B_c) \xrightarrow{\sim} U_{\mathbf{J},<}(B), \qquad \otimes_{c=1}^{\mathrm{C}(\mathbf{J})} U_{\mathbf{J}_c,>}(B_c) \xrightarrow{\sim} U_{\mathbf{J},>}(B). \tag{5.54}$$

(2) *Let C be a real biconvex set in $\Delta_{\mathbf{J}+}$, and y an element of $W_{\mathbf{J}}(C)$. Set $D := y^{-1}\{C \setminus \Phi_{\mathbf{J}}(y)\}$. Then the multiplication defines the following isomorphisms of $\mathbb{Q}(q)$-vector spaces*:

$$U_{\mathbf{J},<}(y) \otimes {}_{\mathbf{J}}T_y U_{\mathbf{J},<}(D) \xrightarrow{\sim} U_{\mathbf{J},<}(C), \tag{5.55}$$

$$ {}_{\mathbf{J}}T_y U_{\mathbf{J},>}(D) \otimes U_{\mathbf{J},>}(y) \xrightarrow{\sim} U_{\mathbf{J},>}(C), \tag{5.56}$$

where $U_{\mathbf{J},<}(y) := U_{\mathbf{J},<}(\Phi_{\mathbf{J}}(y))$ and $U_{\mathbf{J},>}(y) := U_{\mathbf{J},>}(\Phi_{\mathbf{J}}(y))$. In particular, we have $U_{\mathbf{J},<}(y) \subset U_{\mathbf{J},<}(C)$ and $U_{\mathbf{J},>}(y) \subset U_{\mathbf{J},>}(C)$. Moreover, we have:

$$U_{\mathbf{J},<}(y) = U_{\mathbf{J},<}(C) \cap A_{\mathbf{J}}(y), \tag{5.57}$$

$$ {}_{\mathbf{J}}T_y U_{\mathbf{J},<}(D) = U_{\mathbf{J},<}(C) \cap A_{\mathbf{J}}(y)^c, \tag{5.58}$$

$$U_{\mathbf{J},>}(y) = U_{\mathbf{J},>}(C) \cap A_{\mathbf{J}}(y), \tag{5.59}$$

$$ {}_{\mathbf{J}}T_y U_{\mathbf{J},>}(D) = U_{\mathbf{J},>}(C) \cap A_{\mathbf{J}}(y)^c. \tag{5.60}$$

(3) *For each $B \in \mathfrak{B}^*_{\mathbf{J}}$, we have $U_{\mathbf{J},<}(B) \cup U_{\mathbf{J},>}(B) \subset A_{\mathbf{J}}(B)$.*

(4) *The action of $\mathcal{B}_{\widehat{W}_{\mathbf{J}}}$ on $U_{\mathbf{J}}$ is faithful.*

Proof. (1) For each $c = 1, \ldots, \mathrm{C}(\mathbf{J})$, we set $\boldsymbol{s}^{-1}(S_{\mathbf{J}_c}) := \{\, p \in \mathbb{N}_{\ell(\boldsymbol{S})} \mid \boldsymbol{s}(p) \in S_{\mathbf{J}_c} \,\}$ and $n_c := \sharp \boldsymbol{s}^{-1}(S_{\mathbf{J}_c})$, and denote by ι_c the unique strictly increasing function from $\mathbb{N}_{n_c}$ to $\mathbb{N}_{\ell(\boldsymbol{S})}$ such that $\operatorname{Im}\iota_c = \boldsymbol{s}^{-1}(S_{\mathbf{J}_c})$. Then, for each $c = 1, \ldots, \mathrm{C}(\mathbf{J})$, we define a sequence $\boldsymbol{s}_c = (\boldsymbol{s}_c(p))_{p\in\mathbb{N}_{n_c}} \in S_{\mathbf{J}_c}^{\mathbb{N}_{n_c}}$ by setting $\boldsymbol{s}_c(p) := \boldsymbol{s}(\iota_c(p))$ for each $p \in \mathbb{N}_{n_c}$. We see that $\boldsymbol{s}_c$ is an element of $\mathcal{W}^{n_c}_{\mathbf{J}_c}$ such that $\Phi^*_{\mathbf{J}_c}([\boldsymbol{s}_c]) = B_c$. By Proposition 5.19(2), we see that the multiplication defines the following injective $\mathbb{Q}(q)$-linear mappings*:*

$$\otimes_{c=1}^{\mathrm{C}(\mathbf{J})} U_{\mathbf{J}_c,<}(B_c) \hookrightarrow U^+, \qquad \otimes_{c=1}^{\mathrm{C}(\mathbf{J})} U_{\mathbf{J}_c,>}(B_c) \hookrightarrow U^+. \tag{5.61}$$

Moreover, by Lemma 5.4(4) and Proposition 5.10(2), we see that

$$\textstyle\prod_{c=1}^{\mathrm{C}(\mathbf{J})} \mathcal{E}_{\boldsymbol{s}_c,<} = \mathcal{E}_{\boldsymbol{s},<}, \qquad \prod_{c=1}^{\mathrm{C}(\mathbf{J})} \mathcal{E}_{\boldsymbol{s}_c,>} = \mathcal{E}_{\boldsymbol{s},>}. \tag{5.62}$$

Therefore we get the (5.54).

(2) By definition, we see that for each $\boldsymbol{s} \in \mathcal{W}_{\mathbf{J}}^*$ and $m \in \mathbb{N}_{\ell(\boldsymbol{s})}$ the multiplication defines the following isomorphisms of $\mathbb{Q}(q)$-vector spaces*:*

$$U_{\mathbf{J},<}(\boldsymbol{s}_{|m}) \otimes T_{[\boldsymbol{s}_{|m}]} U_{\mathbf{J},<}(\boldsymbol{s}_{|^m}) \xrightarrow{\sim} U_{\mathbf{J},<}(\boldsymbol{s}), \quad a \otimes b \mapsto ab, \tag{5.63}$$

$$T_{[\boldsymbol{s}_{|m}]} U_{\mathbf{J},>}(\boldsymbol{s}_{|^m}) \otimes U_{\mathbf{J},>}(\boldsymbol{s}_{|m}) \xrightarrow{\sim} U_{\mathbf{J},>}(\boldsymbol{s}), \quad a \otimes b \mapsto ab, \tag{5.64}$$

where $\boldsymbol{s}_{|m}$ is the initial m-section of $\boldsymbol{s}$ and $\boldsymbol{s}_{|^m}$ is the m-shift of $\boldsymbol{s}$.

We prove (5.55) and (5.56) in the case where $B = \Phi_{\mathbf{J}}(z) \in \mathfrak{B}_{\mathbf{J}}$ with $z \in W_{\mathbf{J}}$. Since $\Phi_{\mathbf{J}}(y) \subset \Phi_{\mathbf{J}}(z)$, we have $\Phi_{\mathbf{J}}(z) = \Phi_{\mathbf{J}}(y) \amalg y\Phi_{\mathbf{J}}(y^{-1}z)$, and hence $D = \Phi_{\mathbf{J}}(y^{-1}z)$. Thus we see that there exists an element $\boldsymbol{s} \in \mathcal{W}_{\mathbf{J}}^n$ such that $[\boldsymbol{s}] = z$, $[\boldsymbol{s}_{|m}] = y$, and $[\boldsymbol{s}_{|^m}] = y^{-1}z$ with $n = \ell(z)$ and $m = \ell(y)$. Hence, (5.63) and (5.64) imply (5.55) and (5.56).

We prove (5.55) and (5.56) in the case where $B \in \mathfrak{B}^\infty$. By Lemma 2.5 in [8], we see that $D \in \mathfrak{B}_{\mathbf{J}}^\infty$ and $B = \Phi_{\mathbf{J}}(y) \amalg yD$. Thus there exist an element $\boldsymbol{s} \in \mathcal{W}_{\mathbf{J}}^\infty$ such that $\Phi_{\mathbf{J}}^*([\boldsymbol{s}]) = B$, $[\boldsymbol{s}_{|m}] = y$, and $\Phi_{\mathbf{J}}^\infty([\boldsymbol{s}_{|^m}]) = D$ with $m = \ell(y)$. Hence, (5.63) and (5.64) imply (5.55) and (5.56).

It is easy to see that

$$U_{\mathbf{J},<}(y) \cup U_{\mathbf{J},>}(y) \subset A_{\mathbf{J}}(y), \qquad {}_{\mathbf{J}}T_y U_{\mathbf{J},<}(D) \cup {}_{\mathbf{J}}T_y U_{\mathbf{J},>}(D) \subset A_{\mathbf{J}}(y)^c,$$

and hence (5.57)–(5.60) follow from (5.55) and (5.56).

(3) It is easy to see that

$$\mathcal{E}_{\boldsymbol{s},<} = \cup_{p\in\mathbb{N}} \mathcal{E}_{\boldsymbol{s}_{|p},<}, \qquad \mathcal{E}_{\boldsymbol{s},>} = \cup_{p\in\mathbb{N}} \mathcal{E}_{\boldsymbol{s}_{|p},>}$$

with $\mathcal{E}_{\boldsymbol{s}_{|p},<} \subset \mathcal{E}_{\boldsymbol{s}_{|p+1},<}$ and $\mathcal{E}_{\boldsymbol{s}_{|p},>} \subset \mathcal{E}_{\boldsymbol{s}_{|p+1},>}$, which implies that

$$U_{\mathbf{J},<}(B) = \cup_{p\in\mathbb{N}} U_{\mathbf{J},<}([\boldsymbol{s}_{|p}]), \qquad U_{\mathbf{J},>}(B) = \cup_{p\in\mathbb{N}} U_{\mathbf{J},>}([\boldsymbol{s}_{|p}]), \tag{5.65}$$

Thus we get $U_{\mathbf{J},<}(B) \cup U_{\mathbf{J},>}(B) \subset A_{\mathbf{J}}(B)$, since $U_{\mathbf{J},<}([\boldsymbol{s}_{|p}]) \cup U_{\mathbf{J},>}([\boldsymbol{s}_{|p}]) \subset A_{\mathbf{J}}([\boldsymbol{s}_{|p}])$ and $[\boldsymbol{s}_{|p}] \in W_{\mathbf{J}}(B)$ for all $p \in \mathbb{N}$.

(4) Suppose that ${}_{\mathbf{J}}T_y|_{U_{\mathbf{J}}} = id$ for $y \in W_{\mathbf{J}}$. Then $A_{\mathbf{J}}(y) = \mathbb{Q}(q)$. Since $U_{\mathbf{J},<}(y) \subset A_{\mathbf{J}}(y)$, it follows that $U_{\mathbf{J},<}(y) = \mathbb{Q}(q)$. Thus we get $y = 1$ by Proposition 5.19(1). ◻

Definition 5.21. For each $w \in \mathring{W}_{\mathbf{J}}$, we set

$$U_{\mathbf{J},<}(w,-) := U_{\mathbf{J},<}(\Delta_{\mathbf{J}}(w,-)), \quad U_{\mathbf{J},>}(w,-) := U_{\mathbf{J},>}(\Delta_{\mathbf{J}}(w,-)), \tag{5.66}$$

$$U_{\mathbf{J},>}(w,+) := \Psi U_{\mathbf{J},<}(\Delta_{\mathbf{J}}(w,+)), \quad U_{\mathbf{J},<}(w,+) := \Psi U_{\mathbf{J},>}(\Delta_{\mathbf{J}}(w,+)). \tag{5.67}$$

Note that

$$U_{\mathbf{J},>}(w,+) = \Psi U_{\mathbf{J},<}(ww_\circ,-), \qquad U_{\mathbf{J},<}(w,+) = \Psi U_{\mathbf{J},>}(ww_\circ,-) \tag{5.68}$$

with $w_\circ$ the longest element of $\mathring{W}_{\mathbf{J}}$. In the case where $\mathbf{J} = \mathring{\mathbf{I}}$, we will denote the symbols above more simply by removing $\mathbf{J}$ from them.

Proposition 5.22. *For each $w \in \mathring{W}_{\mathbf{J}}$ and $y \in W_{\mathbf{J}}(\Delta_{\mathbf{J}}(w,-))$, the multiplication defines the following isomorphisms of $\mathbb{Q}(q)$-vector spaces*:

$$U_{\mathbf{J},<}(y) \otimes {}_{\mathbf{J}}T_y U_{\mathbf{J},<}(\overline{y}^{-1}w,-) \xrightarrow{\sim} U_{\mathbf{J},<}(w,-), \tag{5.69}$$

$${}_{\mathbf{J}}T_y U_{\mathbf{J},>}(\overline{y}^{-1}w,-) \otimes U_{\mathbf{J},>}(y) \xrightarrow{\sim} U_{\mathbf{J},>}(w,-). \tag{5.70}$$

Moreover, the following equalities hold:

$$U_{\mathbf{J},<}(y) = U_{\mathbf{J},<}(w,-) \cap A_{\mathbf{J}}(y), \tag{5.71}$$

$${}_{\mathbf{J}}T_y U_{\mathbf{J},<}(\overline{y}^{-1}w,-) = U_{\mathbf{J},<}(w,-) \cap A_{\mathbf{J}}(y)^c, \tag{5.72}$$

$$U_{\mathbf{J},>}(y) = U_{\mathbf{J},>}(w,-) \cap A_{\mathbf{J}}(y), \tag{5.73}$$

$${}_{\mathbf{J}}T_y U_{\mathbf{J},>}(\overline{y}^{-1}w,-) = U_{\mathbf{J},>}(w,-) \cap A_{\mathbf{J}}(y)^c. \tag{5.74}$$

Proof. Since $\Phi_{\mathbf{J}}(y) \subset \Delta_{\mathbf{J}}(w,-)$, we have

$$\Phi_{\mathbf{J}}(y) \amalg y\Delta_{\mathbf{J}}(\overline{y}^{-1}w,-) = \Delta_{\mathbf{J}}(w,-). \tag{5.75}$$

Thus the assertions follow from Proposition 5.20(2). □

Lemma 5.23. *Let w be an arbitrary element of $\in \mathring{W}_{\mathbf{J}}$.*

(1) *For each $y \in W_{\mathbf{J}}(\Delta_{\mathbf{J}}(w,-))$, we have*

$${}_{\mathbf{J}}T_y^{-1} U_{\mathbf{J},>}(w,+) \subset U_{\mathbf{J},>}(\overline{y}^{-1}w,+), \quad {}_{\mathbf{J}}T_y^{-1} U_{\mathbf{J},<}(w,+) \subset U_{\mathbf{J},<}(\overline{y}^{-1}w,+).$$

In particular, we have $U_{\mathbf{J},>}(w,+) \cup U_{\mathbf{J},<}(w,+) \subset A_{\mathbf{J}}(w,-)^c$.

(2) *For each $y \in W_{\mathbf{J}}(\Delta_{\mathbf{J}}(w,+))$, we have*

$${}_{\mathbf{J}}T_{y^{-1}} U_{\mathbf{J},<}(w,-) \subset U_{\mathbf{J},<}(\overline{y}^{-1}w,-), \quad {}_{\mathbf{J}}T_{y^{-1}} U_{\mathbf{J},>}(w,-) \subset U_{\mathbf{J},>}(\overline{y}^{-1}w,-).$$

In particular, we have $U_{\mathbf{J},<}(w,-) \cup U_{\mathbf{J},>}(w,-) \subset A_{\mathbf{J}}(w,+)^c$.

(3) *The following* (i) *and* (ii) *hold*:

$$\text{(i)} \quad u \in \Psi T_{w_\circ} U_{\mathbf{J},<}(1,-) \Rightarrow T_w(u) \in U_{\mathbf{J},>}(w,+),$$

$$\text{(ii)} \quad u \in \Psi T_{w_\circ} U_{\mathbf{J},>}(1,-) \Rightarrow T_w(u) \in U_{\mathbf{J},<}(w,+).$$

(4) *The following inclusions hold*:

$$U_{\mathbf{J},<}(w,-) \cup U_{\mathbf{J},>}(w,-) \subset A_{\mathbf{J}}(w,-), \quad U_{\mathbf{J},<}(w,+) \cup U_{\mathbf{J},>}(w,+) \subset A_{\mathbf{J}}(w,+).$$

Proof. (1) By (5.75), we have $\Phi_{\mathbf{J}}(y^{-1}) \subset \Delta_{\mathbf{J}}(\overline{y}^{-1}ww_\circ,-)$. By Proposition 5.22, we have ${}_{\mathbf{J}}T_{y^{-1}} U_{\mathbf{J},<}(ww_\circ,-) \subset U_{\mathbf{J},<}(\overline{y}^{-1}ww_\circ,-)$. Thus, by (5.68), we get

$${}_{\mathbf{J}}T_y^{-1} U_{\mathbf{J},>}(w,+) = \Psi {}_{\mathbf{J}}T_{y^{-1}} U_{\mathbf{J},<}(ww_\circ,-) \subset U_{\mathbf{J},>}(\overline{y}^{-1}w,+).$$

(2) Since $\Phi_{\mathbf{J}}(y) \subset \Delta_{\mathbf{J}}(ww_\circ,-)$ we have $\Phi_{\mathbf{J}}(y^{-1}) \subset \Delta_{\mathbf{J}}(\overline{y}^{-1}w,-)$. Thus, by Proposition 5.20, we get ${}_{\mathbf{J}}T_{y^{-1}} U_{\mathbf{J},<}(w,-) \subset U_{\mathbf{J},<}(\overline{y}^{-1}w,-)$.

(3) By (5.68) and (5.69), we see that the multiplication mapping

$$\Psi U_{\mathbf{J},<}(ww_\circ) \otimes \Psi T_{ww_\circ} U_{\mathbf{J},<}(1,-) \to U_{\mathbf{J},>}(1,+)$$

is an isomorphism of $\mathbb{Q}(q)$-vector spaces. On the other hand, we see that $T_{w_\circ} = T_{w^{-1}} T_{ww_\circ}$ and $T_w \Psi = \Psi T_{w^{-1}}^{-1}$, and hence

$$T_w(u) \in T_w \Psi T_{w_\circ} U_{\mathbf{J},<}(1,-) = \Psi T_{ww_\circ} U_{\mathbf{J},<}(1,-) \subset U_{\mathbf{J},>}(w,+).$$

The proof of (ii) is similar.

(4) The left inclusion follows from Proposition 5.20(3) and (5.66) and the left part of (5.37). The right inclusion follows from Proposition 5.20(3) and (5.67) and the left part of (5.38). □

6. Imaginary root vectors of $U_{\mathbf{J}}^+$

In this section, we introduce imaginary root vectors of $U_{\mathbf{J}}^+$, where $\mathbf{J}$ is an arbitrary non-empty subset of $\mathring{\mathbf{I}}$.

Definition 6.1. For each $(i,m) \in \mathring{\mathbf{I}} \times \mathbb{Z}$, we set

$$x_{i,m}^- := T_{\varepsilon_i}^m T_i^{-1}(E_i), \qquad x_{i,m}^+ := T_{\varepsilon_i}^{-m}(E_i). \tag{6.1}$$

Lemma 6.2. (1) *Suppose that $n \in \mathbb{N}$ and $m \in \mathbb{Z}_+$. Then*

$$T_w(x_{i,n}^-) \in U_{<(w,-)_{n\delta - w(\alpha_i)}}, \tag{6.2}$$

$$T_w(x_{i,m}^+) \in U_{<(w,+)_{m\delta + w(\alpha_i)}} \tag{6.3}$$

for each $i \in \mathring{\mathbf{I}}$ and $w \in \mathring{W}$. Moreover,

$$T_w(x_{i,n}^-) \in {}_{\mathcal{A}_1}U^+ \setminus (q-1)_{\mathcal{A}_1}U^+, \tag{6.4}$$

$$T_w(x_{i,m}^+) \in {}_{\mathcal{A}_1}U^+ \setminus (q-1)_{\mathcal{A}_1}U^+. \tag{6.5}$$

(2) *For each $(j,m) \in \mathbf{J} \times \mathbb{Z}$, we have*

$$x_{j,m}^- = ({}_{\mathbf{J}}T_{\varepsilon_j})^m T_j^{-1}(E_j), \tag{6.6}$$

$$x_{j,m}^+ = ({}_{\mathbf{J}}T_{\varepsilon_j})^{-m}(E_j). \tag{6.7}$$

Therefore, both $x_{j,m}^-$ and $x_{j,m}^+$ are elements of $U_{\mathbf{J}}$. Moreover, we have $x_{j,m}^- \in A_{\mathbf{J}}(1,-)$ if $m > 0$, and $x_{j,m}^+ \in A_{\mathbf{J}}(1,+)$ if $m \geq 0$.

(3) *Let (j,m) be an arbitrary element of $\mathbf{J} \times \mathbb{Z}_+$, and $\boldsymbol{s}$ an arbitrary element of $\mathcal{W}_{\mathbf{J}}$ such that $m\delta + \alpha_j \in \Phi_{\mathbf{J}}([\boldsymbol{s}]) \subset \Delta_{\mathbf{J}}(1,+)$. Then we have $\Psi E_{m\delta+\alpha_j,\boldsymbol{s}} = x_{j,m}^+$.*

Proof. (1) By Definition 5.2, Definition 5.21(5.66), and Definition 6.1, we see that

$$x_{i,1}^- = E_{\delta-\alpha_i} \in U_{<(1,-)_{\delta-\alpha_i}},$$

and hence

$$T_w(x_{i,n}^-) = T_w T_{\varepsilon_i}^{n-1}(E_{\delta-\alpha_i}) \in U_{<(w,-)_{n\delta-w(\alpha_i)}}$$

by (5.69). Let $w_\circ$ be the longest element of $\mathring{W}$. Then, by (5.68)(5.70), we see that the multiplication mapping

$$\Psi U_>(w_\circ) \otimes \Psi T_{w_\circ} U_>(1,-) \to U_<(1,+)$$

is an isomorphism of $\mathbb{Q}(q)$-vector spaces, and hence

$$U_{<(1,+)_{m\delta+\alpha_i}} \subset \Psi T_{w_\circ} U_>(1,-). \tag{6.8}$$

On the other hand, by Lemma 5.23(1), we have

$$x_{i,m}^+ = T_{\varepsilon_i}^{-m}(E_i) \in U_{<(1,+)_{m\delta+\alpha_i}}$$

since $E_i \in U_<(1,+)$. Combining with (6.8), we have

$$x_{i,m}^+ \in U_{<(1,+)_{m\delta+\alpha_i}} \subset \Psi T_{w_\circ} U_>(1,-).$$

Thus, it follows from Lemma 5.23(3)(ii) that

$$T_w(x_{i,m}^+) \in U_{<(w,+)_{m\delta+w(\alpha_i)}}.$$

It is easy to see that the set ${}_{\mathcal{A}_1}U' \setminus (q-1)_{\mathcal{A}_1}U'$ is stable under the action of $\mathcal{B}_{\widehat{W}}$ on U, which implies

$$T_w(x_{i,n}^-), T_w(x_{i,m}^+) \in {}_{\mathcal{A}_1}U' \setminus (q-1)_{\mathcal{A}_1}U'.$$

Moreover, by (6.2)(6.3), we see that $T_w(x^-_{i,n})$ and $T_w(x^+_{i,m})$ are elements of U^+. Thus we get (6.4)(6.5).

(2) Since both E_j and $T_j^{-1}(E_j)$ are elements of $U_{\mathbf{J}}$, the (6.6) and (6.7) follow immediately from (6.1) and Proposition 5.15. Since ${}_{\mathbf{J}}T_{\varepsilon_j}$ is an automorphism of $U_{\mathbf{J}}$, by (6.6) and (6.7) we see that $x^{\pm}_{j,m} \in U_{\mathbf{J}}$. In the case where $m > 0$, by (1) we have $x^-_{j,m} \in U_{\mathbf{J}} \cap U^+ = U^+_{\mathbf{J}}$. In addition, by (6.6) we have

$$({}_{\mathbf{J}}T_{\varepsilon_j})^{-m}(x^-_{j,m}) = -K_j^{-1}F_j \in U_{\mathbf{J}}^{\leq 0},$$

and hence $x^-_{j,m} \in A_{\mathbf{J}}(1,-)$. The proof of remains is similar.

(3) Firstly, we claim that if $(\boldsymbol{s}_1, \boldsymbol{s}_2)$ is a pair of elements of $\mathcal{W}_{\mathbf{J}}$ such that $m\delta+\alpha_j \in \Phi_{\mathbf{J}}([\boldsymbol{s}_i]) \subset \Delta_{\mathbf{J}}(1,+)$ for $i = 1, 2$ then $E_{m\delta+\alpha_j,\boldsymbol{s}_1} = E_{m\delta+\alpha_j,\boldsymbol{s}_2}$. We may assume that $[\boldsymbol{s}_1] = [\boldsymbol{s}_2]$, and put $x = [\boldsymbol{s}_1] = [\boldsymbol{s}_2]$. Since $\gamma \in \Delta_{\mathbf{J}}(1,+)$ for each $\gamma \in \Phi_{\mathbf{J}}(x)$, there exists $\mathbf{d}(\gamma) \in \mathbb{Z}_+$ such that $\gamma = \mathbf{d}(\gamma)\delta + \overline{\gamma}$ with $\overline{\gamma} \in \mathring{\Delta}_{\mathbf{J}+}$. Now suppose that $m\delta + \alpha_j = \sum_{\gamma\in\Phi_{\mathbf{J}}(x)} \mathbf{c}(\gamma)\gamma$ with $\mathbf{c}(\gamma) \in \mathbb{Z}_+$ for all $\gamma \in \Phi_{\mathbf{J}}(x)$. Then $m\delta + \alpha_j = (\sum_{\gamma\in\Phi_{\mathbf{J}}(x)} \mathbf{c}(\gamma)\mathbf{d}(\gamma))\delta + \sum_{\gamma\in\Phi_{\mathbf{J}}(x)} \mathbf{c}(\gamma)\overline{\gamma}$, which implies that $\mathbf{c}(m\delta + \alpha_j) = 1$ and $\mathbf{c}(\gamma) = 0$ for all $\gamma \neq m\delta + \alpha_j$. Thanks to Theorem 5.12 and (5.48), we can apply Lemma 4.5(3) to this case. Thus we get $E_{m\delta+\alpha_j,\boldsymbol{s}_1} = E_{m\delta+\alpha_j,\boldsymbol{s}_2}$.

Now let us prove the equality $\Psi E_{m\delta+\alpha_j,\boldsymbol{s}} = x^+_{j,m}$ for any $\boldsymbol{s} \in \mathcal{W}_{\mathbf{J}}$ such that $m\delta + \alpha_j \in \Phi_{\mathbf{J}}([\boldsymbol{s}]) \subset \Delta_{\mathbf{J}}(1,+)$. Put $l = \ell_{\mathbf{J}}(t_{-m\varepsilon_j})$ and denote $t_{-m\varepsilon_j}$ by $s_1s_2\cdots s_l\rho$ with $s_1, s_2, \ldots, s_l \in S_{\mathbf{J}}$ and $\rho \in \Omega_{\mathbf{J}}$. Let $E_{s_{l+1}} = {}_{\mathbf{J}}T_\rho(E_j)$ with $s_{l+1} \in S_{\mathbf{J}}$, and define $\boldsymbol{s}' = (\boldsymbol{s}'(p))_{p\in\mathbb{N}_{l+1}} \in S_{\mathbf{J}}^{l+1}$ by setting $\boldsymbol{s}'(p) := s_p$ for each $p \in \mathbb{N}_{l+1}$. Then we see that the sequence $\boldsymbol{s}'$ is an element of $\mathcal{W}_{\mathbf{J}}$ satisfying $\Phi_{\mathbf{J}}([\boldsymbol{s}']) \subset \Delta_{\mathbf{J}}(1,+)$ and $\phi_{\boldsymbol{s}'}(l+1) = m\delta + \alpha_j$. Thus it follows from (6.7) and the claim above that

$$\begin{aligned} x^+_{j,m} &= \Psi_{\mathbf{J}}T_{-m\varepsilon_j}(E_j) = \Psi_{\mathbf{J}}T_{s_1} \cdot {}_{\mathbf{J}}T_{s_2} \cdot \cdots \cdot {}_{\mathbf{J}}T_{s_l}(E_{s_{l+1}}) \\ &= \Psi E_{m\delta+\alpha_j,\boldsymbol{s}'} = \Psi E_{m\delta+\alpha_j,\boldsymbol{s}}. \end{aligned}$$

□

Definition 6.3. For each $(i, n) \in \mathring{\mathbf{I}} \times \mathbb{N}$, we set

$$\varphi_{i,n} := [x^-_{i,n}\,,\, E_i]_q = x^-_{i,n}E_i - q_i^{-2}E_ix^-_{i,n}, \tag{6.9}$$

and also define $\varphi_i(z) \in U^+[[z]]$ by setting

$$\varphi_i(z) := (q_i - q_i^{-1}) \sum_{n\geq 1} \varphi_{i,n}z^n. \tag{6.10}$$

In addition, we define $I_{i,n} \in U^+_{n\delta}$ by the following equality in $U^+[[z]]$:

$$I_i(z) = \log(1 + \varphi_i(z)) \equiv \sum_{n\geq 1} \tfrac{(-1)^{n-1}}{n}\varphi_i(z)^n, \tag{6.11}$$

where

$$I_i(z) := (q_i - q_i^{-1}) \sum_{n\geq 1} I_{i,n}z^n. \tag{6.12}$$

Proposition 6.4 ([13]). *For each* $(i,n) \in \overset{\circ}{\mathbf{I}} \times \mathbb{N}$*, the following equalities hold*:

$$I_{i,n} = \sum_{p_1,p_2,\ldots,p_n \geq 0; \sum_k kp_k = n} \frac{(\sum_k p_k - 1)!}{p_1!p_2!\cdots p_n!}(q_i^{-1} - q_i)^{\sum_k p_k - 1}\varphi_{i,1}^{p_1}\varphi_{i,2}^{p_2}\cdots\varphi_{i,n}^{p_n},$$

$$\varphi_{i,n} = \sum_{p_1,p_2,\ldots,p_n \geq 0; \sum_k kp_k = n} \frac{(q_i - q_i^{-1})^{\sum_k p_k - 1}}{p_1!p_2!\cdots p_n!} I_{i,1}^{p_1}I_{i,2}^{p_2}\cdots I_{i,n}^{p_n}.$$

Proposition 6.5 ([1]). *Let* i *and* j *be arbitrary elements of* $\overset{\circ}{\mathbf{I}}$.

(1) *For each* $n \in \mathbb{N}$ *and* $m \in \mathbb{Z}$*, the following equalities hold*:

$$\varphi_{i,n} = [x_{i,m}^-\,,\, x_{i,n-m}^+]_q, \tag{6.13}$$

$$T_{\varepsilon_i}(\varphi_{j,n}) = \varphi_{j,n}, \quad T_{\varepsilon_i}(I_{j,n}) = I_{j,n}, \tag{6.14}$$

$$[x_{j,m}^-\,,\, I_{i,n}] = (\mathrm{sgn}(\mathrm{A}_{ij}))^n \frac{[n\mathrm{A}_{ij}]_{q_i}}{n} x_{j,m+n}^-, \tag{6.15}$$

$$[I_{i,n}\,,\, x_{j,m}^+] = (\mathrm{sgn}(\mathrm{A}_{ij}))^n \frac{[n\mathrm{A}_{ij}]_{q_i}}{n} x_{j,m+n}^+. \tag{6.16}$$

(2) *For each* $m, n \in \mathbb{N}$*, the following equality holds*:

$$[\varphi_{i,m}\,,\, \varphi_{j,n}] = [I_{i,m}\,,\, I_{j,n}] = 0. \tag{6.17}$$

Lemma 6.6. (1) *For each* $w \in \overset{\circ}{W}$ *and* $(i,n) \in \overset{\circ}{\mathbf{I}} \times \mathbb{N}$*, both* $T_w(\varphi_{i,n})$ *and* $T_w(I_{i,n})$ *are elements of* ${}_{\mathcal{A}_1}U_{n\delta}^+ \setminus (q-1){}_{\mathcal{A}_1}U^+$. *Moreover, both* $\overline{T_w(\varphi_{i,n})}$ *and* $\overline{T_w(I_{i,n})}$ *are non-zero elements of* ${}_1U_{n\delta}^+$.

(2) *For each* $w \in \overset{\circ}{W}$*, the elements of* $\{\,\overline{T_w(I_{i,n})}\,|\,(i,n) \in \overset{\circ}{\mathbf{I}} \times \mathbb{N}\,\}$ *are linearly independent over* $\mathbb{Q}$.

(3) *Suppose that* $j \in \mathbf{J} \subset \overset{\circ}{\mathbf{I}}$. *Then both* $T_w(\varphi_{j,n})$ *and* $T_w(I_{j,n})$ *are elements of* $U_{\mathbf{J}}^+$ *for each* $w \in \overset{\circ}{W}_{\mathbf{J}}$ *and* $n \in \mathbb{N}$.

Proof. (1) Suppose that $w(\alpha_i) > 0$. Then we have $T_w(E_i) \in {}_{\mathcal{A}_1}U_{w(\alpha_i)}^+$. Hence, by Lemma 6.2(1), we have $T_w(x_{i,n}^-) \in {}_{\mathcal{A}_1}U_{n\delta - w(\alpha_i)}^+$, and hence

$$T_w(\varphi_{i,n}) = [T_w(x_{i,n}^-)\,,\, T_w(E_i)]_q \in {}_{\mathcal{A}_1}U^+.$$

Suppose that $w(\alpha_i) < 0$. Then we see that $w = w's_i$ and $\ell(w) = \ell(w') + 1$ for some $w' \in \overset{\circ}{W}$, and hence $T_w = T_{w'}T_i$ and $T_{w'}(E_i) \in {}_{\mathcal{A}_1}U^+$. Thus we have

$$T_w(x_{i,0}^-) = T_{w'}T_i(T_i^{-1}(E_i)) = T_{w'}(E_i) \in {}_{\mathcal{A}_1}U^+.$$

Combining with Lemma 6.2(1)(6.5), we get

$$T_w(\varphi_{i,1}) = T_w([x_{i,0}^-\,,\, x_{i,1}^+]_q) = [T_{w'}(E_i)\,,\, T_w(x_{i,1}^+)]_q \in {}_{\mathcal{A}_1}U^+.$$

In the case that $m \geq 2$, by Lemma 6.2(1)(6.4)(6.5), we have

$$T_w(\varphi_{i,n}) = T_w([x_{i,n-1}^-\,,\, x_{i,1}^+]_q) = [T_w(x_{i,n-1}^-)\,,\, T_w(x_{i,1}^+)]_q \in {}_{\mathcal{A}_1}U^+.$$

Therefore we see that $T_w(\varphi_{i,n}) \in {}_{\mathcal{A}_1}U^+$ for each $w \in \overset{\circ}{W}$ and $(i,n) \in \overset{\circ}{\mathbf{I}} \times \mathbb{N}$. By Proposition 6.4, we see that

$$\begin{aligned} T_w(I_{i,n}) &= T_w(\varphi_{i,n}) \\ &\quad + \sum_{\sum_{k=1}^{n-1} kp_k = n} \tfrac{(\sum_k p_k - 1)!}{p_1!p_2!\cdots p_{n-1}!}(q_i^{-1} - q_i)^{\sum_k p_k - 1} T_w(\varphi_{i,1}^{p_1}\varphi_{i,2}^{p_2}\cdots\varphi_{i,n-1}^{p_{n-1}}), \end{aligned}$$

and hence $T_w(I_{i,n}) \in {}_{\mathcal{A}_1}U^+$. Moreover, by (6.18) in Proposition 6.5 and (6.5) in Lemma 6.2(1), we have

$$T_w(I_{i,n}) \in {}_{\mathcal{A}_1}U^+ \setminus (q-1)_{\mathcal{A}_1}U^+,$$

and hence $T_w(\varphi_{i,n}) \in {}_{\mathcal{A}_1}U^+ \setminus (q-1)_{\mathcal{A}_1}U^+$ by (6.14).

(2) We may assume that $w = 1$. Hence, it suffices to show the linear independence over $\mathbb{Q}$ of the elements of $\{\overline{I_{i,n}} \mid i \in \mathring{\mathbf{I}}\}$ for each $n \in \mathbb{N}$. By (6.17) in Proposition 6.5, we see that

$$[\overline{E_{\delta-\alpha_j}}\,,\, \overline{I_{i,n}}] = (\mathrm{sgn}(\mathrm{A}_{ij}))^m \mathrm{A}_{ij}\overline{x^+_{j,n+1}} \tag{6.18}$$

for all $j \in \mathring{\mathbf{I}}$. Now suppose that $\sum_{i=1}^{l} \nu_i \overline{I_{i,n}} = 0$ with $\nu_i \in \mathbb{Q}$. Then (6.20) implies that $\sum_{i=1}^{l} \nu_i (\mathrm{sgn}(\mathrm{A}_{ij}))^n \mathrm{A}_{ij} = 0$ for all $j \in \mathring{\mathbf{I}}$. Thus we have

$$[\nu_1, \ldots, \nu_l][(\mathrm{sgn}(\mathrm{A}_{ij}))^n \mathrm{A}_{ij}]_{i,j\in\mathring{\mathbf{I}}} = [0, \ldots, 0].$$

Since the matrix $[(\mathrm{sgn}(\mathrm{A}_{ij}))^n \mathrm{A}_{ij}]_{i,j\in\mathring{\mathbf{I}}}$ is invertible, we get $[\nu_1, \ldots, \nu_l] = [0, \ldots, 0]$. Therefore the assertion is valid.

(3) By Lemma 6.2(2), we have $x^-_{j,n} \in U^+_{\mathbf{J}}$, and hence

$$\varphi_{j,n} = [x^-_{j,n}\,,\, E_j]_q \in U^+_{\mathbf{J}}.$$

By (6.13) in Proposition 6.4, we have $I_{j,n} \in U^+_{\mathbf{J}}$. Since $U_{\mathbf{J}}$ is stable under the action of T_w, both $T_w(\varphi_{j,n})$ and $T_w(I_{j,n})$ are elements of $U_{\mathbf{J}}$. Thus we get the assertion in (3) by combining with the first assertion in (1). □

Definition 6.7. For each $w \in \mathring{W}_{\mathbf{J}}$, we define a $\mathbb{Q}(q)$-subalgebra $U_{\mathbf{J}}(w,0)$ of $U^+_{\mathbf{J}}$ by setting

$$U_{\mathbf{J}}(w,0) := \langle\, T_w(I_{j,n}) \mid (j,n) \in \mathbf{J} \times \mathbb{N}\,\rangle_{\mathbb{Q}(q)\text{-alg}}. \tag{6.19}$$

Note that Lemma 6.6(3) implies $U_{\mathbf{J}}(w,0) \subset U^+_{\mathbf{J}}$. Let $\preceq$ be an arbitrary total order on the set $\mathbf{J} \times \mathbb{N}$. For each finitely supported function $\mathbf{c}\colon \mathbf{J} \times \mathbb{N} \to \mathbb{Z}_+$, we set

$$I^{\mathbf{c}}_{\prec} := I^{\mathbf{c}(\eta_1)}_{\eta_1} I^{\mathbf{c}(\eta_2)}_{\eta_2} \cdots I^{\mathbf{c}(\eta_m)}_{\eta_m}, \qquad I^{\mathbf{c}}_{\succ} := I^{\mathbf{c}(\eta_m)}_{\eta_m} \cdots I^{\mathbf{c}(\eta_2)}_{\eta_2} I^{\mathbf{c}(\eta_1)}_{\eta_1}, \tag{6.20}$$

where $\mathrm{supp}(\mathbf{c}) = \{\eta_1, \eta_2, \ldots, \eta_m\}$ with $\eta_1 \prec \eta_2 \prec \cdots \prec \eta_m$. Here we set $I^{\mathbf{c}}_{\prec} = I^{\mathbf{c}}_{\succ} := 1$ in the case where $\mathrm{supp}(\mathbf{c}) = \emptyset$. Let us denote by $\mathcal{I}_{\prec}$ (resp. $\mathcal{I}_{\succ}$) the set of all $I^{\mathbf{c}}_{\prec}$ (resp. $I^{\mathbf{c}}_{\succ}$).

Proposition 6.8. *Let w be an arbitrary element of $\mathring{W}_{\mathbf{J}}$, and $\preceq$ an arbitrary total order on $\mathbf{J} \times \mathbb{N}$. Then $U_{\mathbf{J}}(w,0)$ is a commutative $\mathbb{Q}(q)$-subalgebra of $U^+_{\mathbf{J}}$, and the sets $T_w(\mathcal{I}_{\prec})$ and $T_w(\mathcal{I}_{\succ})$, respectively, are bases of $U_{\mathbf{J}}(w,0)$.*

Proof. We may assume that $w = 1$. By Proposition 6.5(2), we see that $U_{\mathbf{J}}(1,0)$ is a commutative $\mathbb{Q}(q)$-subalgebra of $U^+_{\mathbf{J}}$, and hence $\mathcal{I}_{\prec}$ spans $U_{\mathbf{J}}(1,0)$. Thus it suffices to show the linear independence over $\mathbb{Q}(q)$ of the set $\mathcal{I}_{\prec}$. Let us denote by $\overline{\mathcal{I}}_{\prec}$ the image of $\mathcal{I}_{\prec}$ by the specialization at $q = 1$. By Proposition 4.2, Lemma 6.6(2), and the PBW Theorem of Lie algebras over $\mathbb{Q}$, we see that $\overline{\mathcal{I}}_{\prec}$ is a linearly independent set over $\mathbb{Q}$. It follows from Lemma 4.3 that $\mathcal{I}_{\prec}$ is a linearly independent set over $\mathbb{Q}(q)$. The proof of the assertion for $T_w(\mathcal{I}_{\succ})$ is quite similar. □

Lemma 6.9. (1) *For each $t \in \mathrm{T}_{\mathbf{J}}$, there exists an element $t' \in \mathrm{T}_{\mathbf{J}} \cap W_{\mathbf{J}}(\Delta_{\mathbf{J}}(1,-))$ such that $tt' \in \mathrm{T}_{\mathbf{J}} \cap W_{\mathbf{J}}(\Delta_{\mathbf{J}}(1,-))$.*

(2) *Let w be an element of $\overset{\circ}{W}_{\mathbf{J}}$, and y an element of $W_{\mathbf{J}}(\Delta_{\mathbf{J}}(w,-))$. Set $w' = \overline{y}^{-1}w$. Then there exist elements $t, t' \in T_{\mathbf{J}} \cap W_{\mathbf{J}}(\Delta_{\mathbf{J}}(1,-))$ such that $wt = yw't'$ and $\ell_{\mathbf{J}}(wt) = \ell(w) + \ell_{\mathbf{J}}(t) = \ell_{\mathbf{J}}(y) + \ell(w') + \ell_{\mathbf{J}}(t')$.*

Proof. The part (1) is Clear. We prove the part (2). Since $\varPhi_{\mathbf{J}}(y) \subset \Delta_{\mathbf{J}}(w,-)$ we have

$$\varPhi(w) \amalg w\Delta_{\mathbf{J}}(1,-) = \Delta_{\mathbf{J}}(w,-) = \varPhi_{\mathbf{J}}(y) \amalg y\{\varPhi(w') \amalg w'\Delta_{\mathbf{J}}(1,-)\}.$$

Hence we see that $\ell_{\mathbf{J}}(wz) = \ell(w) + \ell_{\mathbf{J}}(z)$ and $\ell_{\mathbf{J}}(yw'z') = \ell_{\mathbf{J}}(y) + \ell(w') + \ell_{\mathbf{J}}(z')$ for all $z, z' \in W_{\mathbf{J}}(\Delta_{\mathbf{J}}(1,-))$. Since $\overline{yw'} = \overline{y}w' = w$, we have $yw' = wt_\mu$ with $\mu \in \overset{\circ}{Q}{}^\vee_{\mathbf{J}}$. By (1), there exists an element $t_\nu \in \mathrm{T}_{\mathbf{J}} \cap W_{\mathbf{J}}(\Delta_{\mathbf{J}}(1,-))$ such that $t_\mu t_\nu \in \mathrm{T}_{\mathbf{J}} \cap W_{\mathbf{J}}(\Delta_{\mathbf{J}}(1,-))$. Set $t = t_\mu t_\nu$ and $t' = t_\nu$. Then $wt = yw't'$. Since $t, t' \in W_{\mathbf{J}}(\Delta_{\mathbf{J}}(1,-))$ we have

$$\ell(w) + \ell_{\mathbf{J}}(t) = \ell_{\mathbf{J}}(wt) = \ell_{\mathbf{J}}(yw't') = \ell_{\mathbf{J}}(y) + \ell(w') + \ell_{\mathbf{J}}(t').$$

□

Proposition 6.10. (1) *For each $t \in \mathrm{T}_{\mathbf{J}}$ and $u \in U_{\mathbf{J}}(1,0)$, we have ${}_{\mathbf{J}}T_t(u) = u$.*

(2) *For each $w \in \overset{\circ}{W}_{\mathbf{J}}$ and $y \in W_{\mathbf{J}}(\Delta_{\mathbf{J}}(w,-))$, we have ${}_{\mathbf{J}}T_y^{-1}U_{\mathbf{J}}(w,0) = U_{\mathbf{J}}(\overline{y}^{-1}w,0)$. In particular, $U_{\mathbf{J}}(w,0) \subset A_{\mathbf{J}}(w,-)^c$.*

(3) *For each $w \in \overset{\circ}{W}_{\mathbf{J}}$ and $y \in W_{\mathbf{J}}(\Delta_{\mathbf{J}}(w,+))$, we have ${}_{\mathbf{J}}T_{y^{-1}}U_{\mathbf{J}}(w,0) = U_{\mathbf{J}}(\overline{y}^{-1}w,0)$. In particular, $U_{\mathbf{J}}(w,0) \subset A_{\mathbf{J}}(w,+)^c$.*

(4) *For each $w \in \overset{\circ}{W}_{\mathbf{J}}$, we have $U_{\mathbf{J}}(w,0) \subset A_{\mathbf{J}}(w,0)$.*

Proof. (1) This follows from Proposition 5.15, (6.16), and Proposition 6.8.

(2) Set $w' = \overline{y}^{-1}w$. By Lemma 6.9(2), we have

$$T_w \cdot {}_{\mathbf{J}}T_t = {}_{\mathbf{J}}T_y \cdot T_{w'} \cdot {}_{\mathbf{J}}T_{t'}$$

for some $t, t' \in \mathrm{T}_{\mathbf{J}} \cap W_{\mathbf{J}}(\Delta_{\mathbf{J}}(1,-))$. By (1), for each $u \in U_{\mathbf{J}}(1,0)$, we have ${}_{\mathbf{J}}T_t(u) = {}_{\mathbf{J}}T_{t'}(u) = u$, and hence

$${}_{\mathbf{J}}T_y^{-1}T_w(u) = T_{w'}(u) \in U_{\mathbf{J}}(\overline{y}^{-1}w,0).$$

Thus the assertion is valid.

(3) Since $\varPhi_{\mathbf{J}}(y) \subset \Delta_{\mathbf{J}}(ww_\circ,-)$, we have

$$\varPhi_{\mathbf{J}}(y) \amalg y\Delta_{\mathbf{J}}(\overline{y}^{-1}ww_\circ,-) = \Delta_{\mathbf{J}}(ww_\circ,-),$$

and hence $\varPhi_{\mathbf{J}}(y^{-1}) \subset \Delta_{\mathbf{J}}(\overline{y}^{-1}w,-)$. By (2), we get $U_{\mathbf{J}}(\overline{y}^{-1}w,0) = {}_{\mathbf{J}}T_{y^{-1}}U_{\mathbf{J}}(w,0)$.

(4) This follows from (5.43), (2), and (3). □

7. Decompositions of $U_{\mathbf{J}}^+$ into tensor products of subalgebras

In this section, we give several decompositions of $U_{\mathbf{J}}^+$ into tensor products of subalgebras, where $\mathbf{J}$ is an arbitrary non-empty subset of $\overset{\circ}{\mathbf{I}}$.

Proposition 7.1. (1) *For each $w \in \overset{\circ}{W}_{\mathbf{J}}$, the multiplication defines the following isomorphisms of $\mathbb{Q}(q)$-vector spaces*:

$$U_{\mathbf{J},<}(w,-) \otimes U_{\mathbf{J}}(w,0) \otimes U_{\mathbf{J},>}(w,+) \xrightarrow{\sim} U_{\mathbf{J}}^+, \tag{7.1}$$

$$U_{\mathbf{J},<}(w,+) \otimes U_{\mathbf{J}}(w,0) \otimes U_{\mathbf{J},>}(w,-) \xrightarrow{\sim} U_{\mathbf{J}}^+. \tag{7.2}$$

Moreover, the following equality holds:

$$U_{\mathbf{J}}(w,0) = A_{\mathbf{J}}(w,0). \tag{7.3}$$

(2) *The multiplication defines the following isomorphism of* $\mathbb{Q}(q)$*-algebras*:

$$\otimes_{c=1}^{\mathrm{C}(\mathbf{J})} U_{\mathbf{J}_c} \xrightarrow{\sim} U_{\mathbf{J}}. \tag{7.4}$$

(3) *The part (1) of Proposition 5.7 is still valid in the case where* $\mathbf{J}$ *is an arbitrary non-empty subset of* $\mathring{\mathbf{I}}$.

Proof. We prove the parts (1) and (2). Let $\boldsymbol{s}$ and $\boldsymbol{s}'$ be elements of $\mathcal{W}_{\mathbf{J}}^{\infty}$ such that $\varPhi_{\mathbf{J}}^{\infty}([\boldsymbol{s}]) = \Delta_{\mathbf{J}}(w,-)$ and $\varPhi_{\mathbf{J}}^{\infty}([\boldsymbol{s}']) = \Delta_{\mathbf{J}}(w,+)$. By Proposition 5.20(1), we see that $\mathcal{E}_{\boldsymbol{S},<}$ and $\varPsi\mathcal{E}_{\boldsymbol{S}',<}$ are bases of $U_{\mathbf{J},<}(w,-)$ and $U_{\mathbf{J},>}(w,+)$ respectively. Let $\preceq$ be a total order on $\mathbf{J}\times\mathbb{N}$. By Proposition 6.8, we see that $T_w(\mathcal{I}_{\prec})$ is a basis of $U(w,0)$. By Proposition 5.19(1), (5.43), the right part of (5.38), and the left part of (5.67), we see that the multiplication define the following injective $\mathbb{Q}(q)$-linear mapping:

$$U_{\mathbf{J},<}(w,-)\otimes A_{\mathbf{J}}(w,0)\otimes\{A_{\mathbf{J}}(w,-)^c\cap U_{\mathbf{J},>}(w,+)\}\hookrightarrow U_{\mathbf{J}}^{+}. \tag{7.5}$$

By Proposition 6.10(3), we have $U_{\mathbf{J}}(w,0)\subset A_{\mathbf{J}}(w,0)$. By Lemma 5.23(1), we have $U_{\mathbf{J},>}(w,+)\subset A_{\mathbf{J}}(w,-)^c$. Thus we see that the multiplication define the following injective $\mathbb{Q}(q)$-linear mapping:

$$m_1\colon U_{\mathbf{J},<}(w,-)\otimes U_{\mathbf{J}}(w,0)\otimes U_{\mathbf{J},>}(w,+)\hookrightarrow U_{\mathbf{J}}^{+}. \tag{7.6}$$

Hence the elements of the subset $\mathcal{E}_{\boldsymbol{S},<}T_w(\mathcal{I}_{\prec})\varPsi(\mathcal{E}_{\boldsymbol{S}',<})$ of $U_{\mathbf{J}}^{+}$ are linearly independent. In the case where $\mathbf{J}$ is connected, by (5.13)(5.14), we see that the set $\mathcal{E}_{\boldsymbol{S},<}T_w(\mathcal{I}_{\prec})\varPsi(\mathcal{E}_{\boldsymbol{S}',<})$ is a basis of $U_{\mathbf{J}}^{+}$, and hence the mapping (7.6) is bijective. To consider the general case, we denote w uniquely by $w=\prod_{c=1}^{\mathrm{C}(\mathbf{J})} w_c$ with $w_c\in\mathring{W}_{\mathbf{J}_c}$. Then the multiplication defines the following isomorphism of $\mathbb{Q}(q)$-vector spaces:

$$U_{\mathbf{J}_c,<}(w_c,-)\otimes U_{\mathbf{J}_c}(w_c,0)\otimes U_{\mathbf{J}_c,>}(w_c,+)\xrightarrow{\sim} U_{\mathbf{J}_c}^{+}. \tag{7.7}$$

for each $c=1,\dots,\mathrm{C}(\mathbf{J})$. By Proposition 5.20(1) and Proposition 6.8, we see that the multiplication defines the following isomorphisms of $\mathbb{Q}(q)$-vector spaces:

$$\otimes_{c=1}^{\mathrm{C}(\mathbf{J})} U_{\mathbf{J}_c,<}(w_c,-)\xrightarrow{\sim} U_{\mathbf{J},<}(w,-), \tag{7.8}$$

$$\otimes_{c=1}^{\mathrm{C}(\mathbf{J})} U_{\mathbf{J}_c}(w_c,0)\xrightarrow{\sim} U_{\mathbf{J}}(w,0), \tag{7.9}$$

$$\otimes_{c=1}^{\mathrm{C}(\mathbf{J})} U_{\mathbf{J}_c,>}(w_c,+)\xrightarrow{\sim} U_{\mathbf{J},>}(w,+). \tag{7.10}$$

Therefore we have the following diagram:

$$\begin{array}{ccc} U_{\mathbf{J},<}(w,-)\otimes U_{\mathbf{J}}(w,0)\otimes U_{\mathbf{J},>}(w,+) & \overset{m_1}{\hookrightarrow} & U_{\mathbf{J}}^{+} \\ \varphi\uparrow & & \uparrow m_2^{+} \\ \otimes_{c=1}^{\mathrm{C}(\mathbf{J})}\big(U_{\mathbf{J}_c,<}(w_c,-)\otimes U_{\mathbf{J}_c}(w_c,0)\otimes U_{\mathbf{J}_c,>}(w_c,+)\big) & \xrightarrow{\sim} & \otimes_{c=1}^{\mathrm{C}(\mathbf{J})} U_{\mathbf{J}_c}^{+}. \end{array}$$

Here, m_2^{+} is defined by the multiplication and φ is defined by setting

$$\varphi(\otimes_{c=1}^{\mathrm{C}(\mathbf{J})} u_c(-)\otimes u_c(0)\otimes u_c(+)) := \Big(\prod_{c=1}^{\mathrm{C}(\mathbf{J})} u_c(-)\Big)\otimes\Big(\prod_{c=1}^{\mathrm{C}(\mathbf{J})} u_c(0)\Big)\otimes\Big(\prod_{c=1}^{\mathrm{C}(\mathbf{J})} u_c(+)\Big),$$

where $u_c(-)\in U_{\mathbf{J}_c,<}(w_c,-)$, $u_c(0)\in U_{\mathbf{J}_c}(w_c,0)$, and $u_c(+)\in U_{\mathbf{J}_c,>}(w_c,+)$. By Lemma 5.4(4), we see that the diagram above is commutative. By (7.8)–(7.10), we see that φ is an isomorphisms of $\mathbb{Q}(q)$-vector spaces, which implies the injectivity of m_2^{+}. By Proposition 5.5, we have the surjectivity of m_2^{+}. Thus both of m_2^{+} and

m_1 are isomorphisms of $\mathbb{Q}(q)$-vector spaces. Moreover it is easy to see that the multiplication defines the following isomorphisms of $\mathbb{Q}(q)$-vector spaces:

$$m_2^-: \otimes_{c=1}^{\mathrm{C}(\mathbf{J})} U^-_{\mathbf{J}_c} \xrightarrow{\sim} U^-_{\mathbf{J}}, \qquad m_2^0: \otimes_{c=1}^{\mathrm{C}(\mathbf{J})} U^0_{\mathbf{J}_c} \xrightarrow{\sim} U^0_{\mathbf{J}}. \tag{7.11}$$

Thus, by Proposition 5.7(2), we see that the multiplication defines the following isomorphism of $\mathbb{Q}(q)$-vector spaces:

$$m_2: \otimes_{c=1}^{\mathrm{C}(\mathbf{J})} U_{\mathbf{J}_c} \xrightarrow{\sim} U_{\mathbf{J}}. \tag{7.12}$$

It is easy to see that m_2 is compatible with the standard $\mathbb{Q}(q)$-algebra structure of the tensor product $\otimes_{c=1}^{\mathrm{C}(\mathbf{J})} U_{\mathbf{J}_c}$. The equality (7.3) follows from Lemma 6.10(4), (7.1), (5.43), and Lemma 5.23(4).

We prove the part (3). The characterization of $U_{\mathbf{J}}$ in terms of the generators and the defining relations follows from the part (2) and the first assertion of Proposition 5.7(1). The equalities (5.13) and (5.14) follow from the part (1) and Proposition 6.8. ◻

Proposition 7.2. (1) *For each $y \in W_{\mathbf{J}}$, the multiplication defines the following isomorphisms of $\mathbb{Q}(q)$-vector spaces*:

$$A_{\mathbf{J}}(y) \otimes A_{\mathbf{J}}(y)^c \xrightarrow{\sim} U^+_{\mathbf{J}}, \tag{7.13}$$

$$A_{\mathbf{J}}(y)^c \otimes A_{\mathbf{J}}(y) \xrightarrow{\sim} U^+_{\mathbf{J}}. \tag{7.14}$$

(2) *Let B be an arbitrary real biconvex set in $\Delta_{\mathbf{J}+}$. Then the equality holds*:

$$U_{\mathbf{J},<}(B) = A_{\mathbf{J}}(B) = U_{\mathbf{J},>}(B). \tag{7.15}$$

Moreover, $A_{\mathbf{J}}(B)$ is a $\mathbb{Q}(q)$-subalgebra of $U^+_{\mathbf{J}}$.

Proof. We prove (7.13). We may assume that $\varPhi_{\mathbf{J}}(y) \subset \Delta_{\mathbf{J}}(w,-)$ for some $w \in \overset{\circ}{W}_{\mathbf{J}}$. Then there exits an element $\boldsymbol{s} \in \mathcal{W}^\infty_{\mathbf{J}}$ such that $\varPhi^\infty_{\mathbf{J}}([\boldsymbol{s}]) = \Delta_{\mathbf{J}}(w,-)$ and $[\boldsymbol{s}_{|p}] = y$ with $p = \ell(y)$. Let $\boldsymbol{s}'$ be an element of $\mathcal{W}^*_{\mathbf{J}}$ such that $\varPhi_{\mathbf{J}}([\boldsymbol{s}']) = \Delta_{\mathbf{J}}(w,+)$, and $\preceq$ a total order on $\mathbf{J} \times \mathbb{N}$. Then the product set $\mathcal{E}_{\boldsymbol{S},<} T_w(\mathcal{I}_{\prec}) \varPsi(\mathcal{E}_{\boldsymbol{S}',<})$ is a basis of $U^+_{\mathbf{J}}$. From Lemma 5.17, it follows that the multiplication $A_{\mathbf{J}}(y) \otimes A_{\mathbf{J}}(y)^c \to U^+_{\mathbf{J}}$ is injective. Moreover, we see that

$$\mathcal{E}_{\boldsymbol{S}_{|p},<} \subset A_{\mathbf{J}}(y), \quad \mathcal{E}_{\boldsymbol{S}^{|p},<} T_w(\mathcal{I}_{\prec}) \varPsi \mathcal{E}_{\boldsymbol{S}',<} \subset A_{\mathbf{J}}(y)^c, \quad \mathcal{E}_{\boldsymbol{S}_{|p},<}\, {}_{\mathbf{J}}T_y(\mathcal{E}_{\boldsymbol{S}^{|p},<}) = \mathcal{E}_{\boldsymbol{S},<}.$$

Therefore we see that the multiplication $A_{\mathbf{J}}(y) \otimes A_{\mathbf{J}}(y)^c \to U^+_{\mathbf{J}}$ is bijective and that the sets $\mathcal{E}_{\boldsymbol{S}_{|p},<}$ and ${}_{\mathbf{J}}T_y(\mathcal{E}_{\boldsymbol{S}^{|p},<}) T_w(\mathcal{I}_{\prec}) \varPsi(\mathcal{E}_{\boldsymbol{S}',<})$ are bases of $A_{\mathbf{J}}(y)$ and $A_{\mathbf{J}}(y)^c$, respectively. Since $\mathcal{E}_{\boldsymbol{S}_{|p},<}$ is also a basis of $U_{\mathbf{J},<}(y)$, we get $U_{\mathbf{J},<}(y) = A_{\mathbf{J}}(y)$. Similarly, we can prove (7.15) and the equality $U_{\mathbf{J},>}(y) = A_{\mathbf{J}}(y)$. Hence (7.14) is proved in the case where $B = \varPhi_{\mathbf{J}}(y)$ with $y \in W_{\mathbf{J}}$.

We prove (7.15) in the case where $B \in \mathfrak{B}^\infty_{\mathbf{J}}$. Suppose that $\varPhi^\infty_{\mathbf{J}}([\boldsymbol{s}]) = B$ with $\boldsymbol{s} \in \mathcal{W}^\infty_{\mathbf{J}}$. Then $U_{\mathbf{J},<}([\boldsymbol{s}_{|p}]) = A_{\mathbf{J}}([\boldsymbol{s}_{|p}]) = U_{\mathbf{J},>}([\boldsymbol{s}_{|p}])$ for all $p \in \mathbb{N}$. Since $U_{\mathbf{J},<}(B) = \cup_{p\in\mathbb{N}} U_{\mathbf{J},<}([\boldsymbol{s}_{|p}])$ and $U_{\mathbf{J},>}(B) = \cup_{p\in\mathbb{N}} U_{\mathbf{J},>}([\boldsymbol{s}_{|p}])$, we have $U_{\mathbf{J},<}(B) = U_{\mathbf{J},>}(B) = \cup_{p\in\mathbb{N}} A_{\mathbf{J}}([\boldsymbol{s}_{|p}])$. It follows that $U_{\mathbf{J},<}(B) = U_{\mathbf{J},>}(B) \subset A_{\mathbf{J}}(B)$, since $A_{\mathbf{J}}([\boldsymbol{s}_{|p}]) \subset A_{\mathbf{J}}(B)$ for all $p \in \mathbb{N}$. Let y be an arbitrary element of $W_{\mathbf{J}}(B)$. Then we have $A_{\mathbf{J}}(y) = U_{\mathbf{J},<}(y) \subset U_{\mathbf{J},<}(B)$. Thus we get $A_{\mathbf{J}}(B) \subset U_{\mathbf{J},<}(B) = U_{\mathbf{J},>}(B)$. Therefore (7.15) is valid.

We prove the second assertion of the part (2). Suppose that u_1 and u_2 are elements of $A_{\mathbf{J}}(B)$. By (5.36), we may assume that $u_i \in A_{\mathbf{J}}(y_i)$ with $y_i \in W_{\mathbf{J}}(B)$ for $i = 1, 2$. By Lemma 3.7(1), there exists an elements $y_3 \in W_{\mathbf{J}}(B)$ such that $\varPhi_{\mathbf{J}}(y_1) \cup \varPhi_{\mathbf{J}}(y_2) \subset \varPhi_{\mathbf{J}}(y_3)$. By Proposition 5.20(2) and the equality $U_{\mathbf{J},<}(y) = A(y)$

for $y \in W$, we see that $A_{\mathbf{J}}(y_3)$ is a $\mathbb{Q}(q)$-subalgebra of $U_{\mathbf{J}}^{+}$ such that $A_{\mathbf{J}}(y_1) \cup A_{\mathbf{J}}(y_2) \subset A_{\mathbf{J}}(y_3)$, and hence $u_1 + u_2, u_1u_2 \in A_{\mathbf{J}}(y_3) \subset A_{\mathbf{J}}(B)$. Therefore $A_{\mathbf{J}}(B)$ is a $\mathbb{Q}(q)$-subalgebra of $U_{\mathbf{J}}^{+}$. ☐

Proposition 7.3. *For each $w \in \overset{\circ}{W}_{\mathbf{J}}$, the multiplication defines the following isomorphisms of vector spaces*:

$$A_{\mathbf{J}}(w,-) \otimes A_{\mathbf{J}}(w,0) \otimes A_{\mathbf{J}}(w,+) \xrightarrow{\sim} U_{\mathbf{J}}^{+} \xleftarrow{\sim} A_{\mathbf{J}}(w,+) \otimes A_{\mathbf{J}}(w,0) \otimes A_{\mathbf{J}}(w,-), \tag{7.16}$$

$$A_{\mathbf{J}}(w,0) \otimes A_{\mathbf{J}}(w,+) \xrightarrow{\sim} A_{\mathbf{J}}(w,-)^{c} \xleftarrow{\sim} A_{\mathbf{J}}(w,+) \otimes A_{\mathbf{J}}(w,0), \tag{7.17}$$

$$A_{\mathbf{J}}(w,-) \otimes A_{\mathbf{J}}(w,0) \xrightarrow{\sim} A_{\mathbf{J}}(w,+)^{c} \xleftarrow{\sim} A_{\mathbf{J}}(w,0) \otimes A_{\mathbf{J}}(w,-), \tag{7.18}$$

$$A_{\mathbf{J}}(w,-) \otimes A_{\mathbf{J}}(w,-)^{c} \xrightarrow{\sim} U_{\mathbf{J}}^{+} \xleftarrow{\sim} A_{\mathbf{J}}(w,-)^{c} \otimes A_{\mathbf{J}}(w,-). \tag{7.19}$$

Proof. The isomorphism (7.16) follows from (7.1), (7.3), and (7.15). The isomorphism (7.17) follows from (7.16), Lemma 5.23(1), and Proposition 6.10(2). The isomorphism (7.18) follows from (7.16), Lemma 5.23(2), and Proposition 6.10(3). The isomorphism (7.19) follows from (7.16)–(7.18). ☐

Proposition 7.4. *Let B be an arbitrary real biconvex set in $\Delta_{\mathbf{J}+}$.*

(1) *Suppose that B_1 is a real biconvex set in $\Delta_{\mathbf{J}+}$ such that $B \subset B_1$. Then the multiplication defines the following isomorphisms of $\mathbb{Q}(q)$-vector spaces*:

$$A_{\mathbf{J}}(B) \otimes \{A_{\mathbf{J}}(B)^{c} \cap A_{\mathbf{J}}(B_1)\} \xrightarrow{\sim} A_{\mathbf{J}}(B_1), \tag{7.20}$$

$$\{A_{\mathbf{J}}(B)^{c} \cap A_{\mathbf{J}}(B_1)\} \otimes A_{\mathbf{J}}(B) \xrightarrow{\sim} A_{\mathbf{J}}(B_1), \tag{7.21}$$

$$\{A_{\mathbf{J}}(B)^{c} \cap A_{\mathbf{J}}(B_1)\} \otimes A_{\mathbf{J}}(B_1)^{c} \xrightarrow{\sim} A_{\mathbf{J}}(B)^{c}, \tag{7.22}$$

$$A_{\mathbf{J}}(B_1)^{c} \otimes \{A_{\mathbf{J}}(B)^{c} \cap A_{\mathbf{J}}(B_1)\} \xrightarrow{\sim} A_{\mathbf{J}}(B)^{c}. \tag{7.23}$$

(2) *The multiplication defines the following isomorphisms of $\mathbb{Q}(q)$-vector spaces*:

$$A_{\mathbf{J}}(B) \otimes A_{\mathbf{J}}(B)^{c} \xrightarrow{\sim} U_{\mathbf{J}}^{+}, \tag{7.24}$$

$$A_{\mathbf{J}}(B)^{c} \otimes A_{\mathbf{J}}(B) \xrightarrow{\sim} U_{\mathbf{J}}^{+}. \tag{7.25}$$

Proof. We prove (7.20). By Proposition 5.19(1) and (7.15), we see that the multiplication defines the following injective $\mathbb{Q}(q)$-linear mapping:

$$m_1 \colon A_{\mathbf{J}}(B) \otimes A_{\mathbf{J}}(B)^{c} \hookrightarrow U_{\mathbf{J}}^{+}. \tag{7.26}$$

Since $A_{\mathbf{J}}(B) \subset A_{\mathbf{J}}(B_1)$, we see that the multiplication defines the following injective $\mathbb{Q}(q)$-linear mapping:

$$A_{\mathbf{J}}(B) \otimes \{A_{\mathbf{J}}(B)^{c} \cap A_{\mathbf{J}}(B_1)\} \hookrightarrow A_{\mathbf{J}}(B_1). \tag{7.27}$$

Hence it suffices to show the surjectivity of (7.27). Suppose that $B \in \mathfrak{B}_{\mathbf{J}}$ and $B = \Phi_{\mathbf{J}}(y)$ with $y \in W_{\mathbf{J}}$. By Proposition 5.18(2) and (7.15), we see that

$$A_{\mathbf{J}}(B) \otimes {}_{\mathbf{J}}T_y A_{\mathbf{J}}(B') \xrightarrow{\sim} A_{\mathbf{J}}(B_1), \tag{7.28}$$

where $B' = y^{-1}\{B_1 \setminus B\}$. It is clear that ${}_{\mathbf{J}}T_y A_{\mathbf{J}}(B') \subset A_{\mathbf{J}}(B)^{c} \cap A_{\mathbf{J}}(B_1)$. By (7.28), we see that the mapping (7.27) is surjective and ${}_{\mathbf{J}}T_y A_{\mathbf{J}}(B') = A_{\mathbf{J}}(B)^{c} \cap A_{\mathbf{J}}(B_1)$ in this case. Therefore we see that if $y \in W_{\mathbf{J}}(B_1)$ then the multiplication defines the following isomorphism of $\mathbb{Q}(q)$-vector spaces:

$$A_{\mathbf{J}}(y) \otimes \{A_{\mathbf{J}}(y)^{c} \cap A_{\mathbf{J}}(B_1)\} \xrightarrow{\sim} A_{\mathbf{J}}(B_1). \tag{7.29}$$

We next suppose that $B \in \mathfrak{B}_{\mathbf{J}}^{\infty}$ and $\Phi_{\mathbf{J}}^{\infty}([\boldsymbol{s}]) = B$ with $\boldsymbol{s} \in \mathcal{W}_{\mathbf{J}}^{\infty}$. Let u be an arbitrary weight vector of $A_{\mathbf{J}}(B_1)$ with weight β. By (7.13), we see that u can be written as

$$u = \sum_{\lambda \in \Lambda_1} X(1)_\lambda Y(1)_\lambda$$

with $X(1)_\lambda$ and $Y(1)_\lambda$ are weight vectors of $A_{\mathbf{J}}([\boldsymbol{s}|_1])$ and $A_{\mathbf{J}}([\boldsymbol{s}|_1])^c \cap A_{\mathbf{J}}(B_1)$ respectively. Similarly, we see that for each $\lambda \in \Lambda_1$, the weight vector $Y(1)_\lambda$ can be written as

$$Y(1)_\lambda = \sum_{\mu \in \Lambda_2} X(2)_\mu Y(2)_\mu$$

with $X(2)_\mu$ and $Y(2)_\mu$ are weight vectors of $A_{\mathbf{J}}([\boldsymbol{s}|_2)$ and $A_{\mathbf{J}}([\boldsymbol{s}|_2)^c \cap A_{\mathbf{J}}(B_1)$ respectively. Applying the procedure above recursively, we see that for each $p \geq 2$ and $\lambda \in \Lambda_{p-1}$, the weight vector $Y(p-1)_\lambda$ can be written as

$$Y(p-1)_\lambda = \sum_{\mu \in \Lambda_p} X(p)_\mu Y(p)_\mu \tag{7.30}$$

with $X(p)_\mu$ and $Y(p)_\mu$ are weight vectors of $A_{\mathbf{J}}([\boldsymbol{s}|_p])$ and $A_{\mathbf{J}}([\boldsymbol{s}|_p])^c \cap A_{\mathbf{J}}(B_1)$ respectively. Then we have

$$u = \sum_{\lambda_1 \in \Lambda_1, \ldots, \lambda_p \in \Lambda_p} X(1)_{\lambda_1} \cdots X(p)_{\lambda_p} Y(p)_{\lambda_p}, \tag{7.31}$$

and hence

$$\beta = \mathrm{wt}(X(1)_{\lambda_1}) + \cdots + \mathrm{wt}(X(p)_{\lambda_p}) + \mathrm{wt}(Y(p)_{\lambda_p})$$

for all $\lambda_1 \in \Lambda_1, \ldots, \lambda_p \in \Lambda_p$. Thus there exists $p_0 \in \mathbb{N}$ such that $\mathrm{wt}(X(p)_\lambda) = 0$ for all $p \geq p_0 + 1$ and $\lambda \in \Lambda_p$. Hence we may assume that $X(p)_\lambda = 1$ for all $p \geq p_0 + 1$ and $\lambda \in \Lambda_p$. By (7.30), we see that

$$Y(p-1)_\lambda = \sum_{\mu \in \Lambda_p} Y(p)_\mu \in A_{\mathbf{J}}([\boldsymbol{s}|_p])^c \cap A_{\mathbf{J}}(B_1)$$

for all $p \geq p_0 + 1$ and $\lambda \in \Lambda_{p-1}$, which implies that

$$Y(p_0)_\lambda \in \cap_{p \geq p_0} \{A_{\mathbf{J}}([\boldsymbol{s}|_p])^c \cap A_{\mathbf{J}}(B_1)\} = A_{\mathbf{J}}(B)^c \cap A_{\mathbf{J}}(B_1)$$

for all $\lambda \in \Lambda_{p_0}$. Since $A_{\mathbf{J}}([\boldsymbol{s}|_{p_0}])$ is a subalgebra of $A_{\mathbf{J}}(B)$ such that $A_{\mathbf{J}}([\boldsymbol{s}|_p]) \subset A_{\mathbf{J}}([\boldsymbol{s}|_{p_0}])$ for all $p \leq p_0$, we have

$$X(1)_{\lambda_1} \cdots X(p_0)_{\lambda_{p_0}} \in A_{\mathbf{J}}([\boldsymbol{s}|_{p_0}]) \subset A_{\mathbf{J}}(B).$$

By (7.31), we have

$$u = \sum_{\lambda_1 \in \Lambda_1, \ldots, \lambda_{p_0} \in \Lambda_{p_0}} X(1)_{\lambda_1} \cdots X(p_0)_{\lambda_{p_0}} Y(p_0)_{\lambda_{p_0}}.$$

Therefore (7.27) is surjective. The proof of (7.21) is quite similar.

We prove (7.24). Since m_1 is injective, it suffices to show the surjectivity of m_1. By Proposition 6.3(1) in [7], we may assume that $B \subset \Delta_{\mathbf{J}}(w, -)$ for some $w \in \mathring{W}_{\mathbf{J}}$. Then we have $A_{\mathbf{J}}(w,-)^c \subset A_{\mathbf{J}}(B)^c$. Hence, by (7.19), we see that the multiplication defines the following injective $\mathbb{Q}(q)$-linear mapping:

$$m_2 \colon \{A_{\mathbf{J}}(B)^c \cap A_{\mathbf{J}}(w,-)\} \otimes A_{\mathbf{J}}(w,-)^c \hookrightarrow A_{\mathbf{J}}(B)^c.$$

On the other hand, by (7.20), we see that the multiplication defines the following isomorphism of $\mathbb{Q}(q)$-vector spaces:

$$A_{\mathbf{J}}(B) \otimes \{A_{\mathbf{J}}(B)^c \cap A_{\mathbf{J}}(w,-)\} \xrightarrow{\sim} A_{\mathbf{J}}(w,-). \tag{7.32}$$

From (7.32) and (7.19), it follows that the multiplication defines the following isomorphism of $\mathbb{Q}(q)$-vector spaces:

$$m_3\colon A_{\mathbf{J}}(B)\otimes\{A_{\mathbf{J}}(B)^c\cap A_{\mathbf{J}}(w,-)\}\otimes A_{\mathbf{J}}(w,-)^c\xrightarrow{\sim}U_{\mathbf{J}}^+. \tag{7.33}$$

Since $m_3=m_1\circ(id\otimes m_2)$, we see that m_1 is surjective, and hence m_1 is bijective. The proof of (7.25) is quite similar.

We prove (7.22). Since $B\subset B_1$, we have $A_{\mathbf{J}}(B_1)^c\subset A_{\mathbf{J}}(B)^c$. Hence the multiplication define the following injective $\mathbb{Q}(q)$-linear mapping:

$$m_4\colon \{A_{\mathbf{J}}(B)^c\cap A_{\mathbf{J}}(B_1)\}\otimes A_{\mathbf{J}}(B_1)^c\hookrightarrow A_{\mathbf{J}}(B)^c.$$

On the other hand, by (7.20) and (7.24), we see that the multiplication defines the following isomorphism of $\mathbb{Q}(q)$-vector spaces:

$$m_5\colon A_{\mathbf{J}}(B)\otimes\{A_{\mathbf{J}}(B)^c\cap A_{\mathbf{J}}(B_1)\}\otimes A_{\mathbf{J}}(B_1)^c\xrightarrow{\sim}U_{\mathbf{J}}^+. \tag{7.34}$$

Since $m_5=m_1\circ(id\otimes m_4)$ with m_1 is bijective, we see that $id\otimes m_4$ is bijective, and hence m_4 is bijective. The proof of (7.23) is similar. □

Lemma 7.5. *Let y be an arbitrary element of $W_{\mathbf{J}}$, $\boldsymbol{s}$ an element of $\mathcal{W}_{\mathbf{J}}$ such that $[\boldsymbol{s}]=y$, and ε an element of $\mathring{P}^\vee$ such that $(\varepsilon\,|\,\alpha_i)>0$ for all $i\in\mathring{\mathbf{I}}\setminus\mathbf{J}$ and $(\varepsilon\,|\,\alpha_j)=0$ for all $j\in\mathbf{J}$. Then, for each $n\in\mathbb{Z}_{\geq 0}$, we have*

$$A([\widetilde{\boldsymbol{s}}]t_\varepsilon^n)\cap U_{\mathbf{J}}^+=A_{\mathbf{J}}(y), \tag{7.35}$$

$$A([\widetilde{\boldsymbol{s}}]t_\varepsilon^n)^c\cap U_{\mathbf{J}}^+=A_{\mathbf{J}}(y)^c. \tag{7.36}$$

Proof. By the definitions of the action of ${}_{\mathbf{J}}T_y$ on $U_{\mathbf{J}}$ and the subalgebra $A([\widetilde{\boldsymbol{s}}]\,t_\varepsilon^n)$ of U^+, we have

$$\begin{aligned}A([\widetilde{\boldsymbol{s}}]t_\varepsilon^n)\cap U_{\mathbf{J}}^+&=\{\,u\in U_{\mathbf{J}}^+\mid T_\varepsilon^{-n}\,{}_{\mathbf{J}}T_y^{-1}(u)\in U^{\leq 0}\,\}\\&=\{\,u\in U_{\mathbf{J}}^+\mid {}_{\mathbf{J}}T_y^{-1}(u)\in U_{\mathbf{J}}^{\leq 0}\,\}=A_{\mathbf{J}}(y),\end{aligned}$$

where the second equality follows from Lemma 5.4(3)(i) and Lemma 5.14(1). The proof of (7.36) is similar. □

Proposition 7.6. *Let B be a real biconvex set in $\Delta_{\mathbf{J}+}$, and set*

$$\widetilde{B}:=B\amalg\Delta^{\mathbf{J}}(1,-). \tag{7.37}$$

Then $\widetilde{B}$ is a real biconvex set and the following equalities hold:

$$A_{\mathbf{J}}(B)=A(\widetilde{B})\cap U_{\mathbf{J}}^+, \tag{7.38}$$

$$A_{\mathbf{J}}(B)^c=A(\widetilde{B})^c\cap U_{\mathbf{J}}^+. \tag{7.39}$$

Proof. We may assume that $\mathbf{J}\subsetneq\mathring{\mathbf{I}}$. Suppose that $B\in\mathfrak{B}_{\mathbf{J}}$. Then $B=\Phi_{\mathbf{J}}(y)$ for some $y\in W_{\mathbf{J}}$. Let $\boldsymbol{s}$ be an element of $\mathcal{W}_{\mathbf{J}}$ such that $[\boldsymbol{s}]=y$, and ε an element of $\mathring{P}^\vee$ such that $(\varepsilon\,|\,\alpha_i)>0$ for all $i\in\mathring{\mathbf{I}}\setminus\mathbf{J}$ and $(\varepsilon\,|\,\alpha_j)=0$ for all $j\in\mathbf{J}$. By Proposition 3.8, we have $\cup_{n\geq 0}\Phi([\widetilde{\boldsymbol{s}}]t_\varepsilon^n)=\widetilde{B}$, hence $\widetilde{B}\in\mathfrak{B}^\infty$ by Lemma 3.7(2). Thus, by Lemma 7.5, we have

$$A(\widetilde{B})\cap U_{\mathbf{J}}^+=\{\bigcup_{n\geq 0}A([\widetilde{\boldsymbol{s}}]t_\varepsilon^n)\}\cap U_{\mathbf{J}}^+=\bigcup_{n\geq 0}\{A([\widetilde{\boldsymbol{s}}]t_\varepsilon^n)\cap U_{\mathbf{J}}^+\}=A_{\mathbf{J}}(y)=A_{\mathbf{J}}(B),$$

$$A(\widetilde{B})^c\cap U_{\mathbf{J}}^+=\{\bigcap_{n\geq 0}A([\widetilde{\boldsymbol{s}}]t_\varepsilon^n)^c\}\cap U_{\mathbf{J}}^+=\bigcap_{n\geq 0}\{A([\widetilde{\boldsymbol{s}}]t_\varepsilon^n)^c\cap U_{\mathbf{J}}^+\}=A_{\mathbf{J}}(y)^c=A_{\mathbf{J}}(B)^c.$$

Suppose that $B \in \mathfrak{B}^{\infty}_{\mathbf{J}}$. Let $\boldsymbol{s}$ be an element of $\mathcal{W}^{\infty}_{\mathbf{J}}$ such that $\Phi^{\infty}_{\mathbf{J}}([\boldsymbol{s}]) = B$. By the definitions of $A_{\mathbf{J}}([\boldsymbol{s}_{|p}])$ and $A_{\mathbf{J}}([\boldsymbol{s}_{|p}])^c$, for each $p \in \mathbb{N}$, we have

$$A_{\mathbf{J}}([\boldsymbol{s}_{|p}]) = A([\widetilde{\boldsymbol{s}_{|p}}]) \cap U^+_{\mathbf{J}},$$
$$A_{\mathbf{J}}([\boldsymbol{s}_{|p}])^c = A([\widetilde{\boldsymbol{s}_{|p}}])^c \cap U^+_{\mathbf{J}}.$$

By Lemma 3.6(2), we have $\cup_{p\in\mathbb{N}}\Phi([\widetilde{\boldsymbol{s}_{|p}}]) = \widetilde{B}$, and hence $\widetilde{B} \in \mathfrak{B}^{\infty}$ by Lemma 3.7(2). Thus we get

$$A_{\mathbf{J}}(B) = \bigcup_{p\in\mathbb{N}} A_{\mathbf{J}}([\boldsymbol{s}_{|p}]) = \{\bigcup_{p\in\mathbb{N}} A([\widetilde{\boldsymbol{s}_{|p}}])\} \cap U^+_{\mathbf{J}} = A(\widetilde{B}) \cap U^+_{\mathbf{J}},$$
$$A_{\mathbf{J}}(B)^c = \bigcap_{p\in\mathbb{N}} A_{\mathbf{J}}([\boldsymbol{s}_{|p}])^c = \{\bigcap_{p\in\mathbb{N}} A([\widetilde{\boldsymbol{s}_{|p}}])^c\} \cap U^+_{\mathbf{J}} = A(\widetilde{B})^c \cap U^+_{\mathbf{J}}.$$

□

8. Convex bases of $U^+_{\mathbf{J}}$

The aim of this section is to construct convex bases of $U^+_{\mathbf{J}}$ associated with all convex orders on $\Delta_{\mathbf{J}+}$, where $\mathbf{J}$ is an arbitrary non-empty subset of $\mathring{\mathbf{I}}$.

Proposition 8.1. *Let C_1 and C_2 be real biconvex sets in $\Delta_{\mathbf{J}+}$ such that $C_1 \subset C_2$. Denote C_1 and C_2 uniquely by $C_1 = \nabla_{\mathbf{J}}(\mathbf{K}, w, y)$ and $C_2 = C_1 \amalg wyB$ with $\mathbf{K} \subset \mathbf{J}$, $w \in \mathring{W}^{\mathbf{K}}_{\mathbf{J}}$, $y \in W_{\mathbf{K}}$, and $B \in \mathfrak{B}^*_{\mathbf{K}}$. Then the following equality holds*:

$$A_{\mathbf{J}}(C_1)^c \cap A_{\mathbf{J}}(C_2) = T_w \cdot {}_{\mathbf{J}}T_y A_{\mathbf{K}}(B). \tag{8.1}$$

Moreover, the multiplication defines the following isomorphisms of $\mathbb{Q}(q)$-vector spaces:

$$A_{\mathbf{J}}(C_1) \otimes T_w \cdot {}_{\mathbf{J}}T_y A_{\mathbf{K}}(B) \xrightarrow{\sim} A_{\mathbf{J}}(C_2), \tag{8.2}$$
$$T_w \cdot {}_{\mathbf{J}}T_y A_{\mathbf{K}}(B) \otimes A_{\mathbf{J}}(C_1) \xrightarrow{\sim} A_{\mathbf{J}}(C_2). \tag{8.3}$$

Proof. In the case where $C_1 \in \mathfrak{B}_{\mathbf{J}}$, we have $\mathbf{K} = \mathbf{J}$ and $C_1 = \Phi_{\mathbf{J}}(y)$, which implies that $w = 1$ and $B = y^{-1}\{C_1 \setminus \Phi_{\mathbf{J}}(y)\}$. Thus $A_{\mathbf{J}}(C_1)^c \cap A_{\mathbf{J}}(C_2) = {}_{\mathbf{J}}T_y A_{\mathbf{J}}(B)$ by Proposition 5.20(2) and Proposition 7.2(2). Therefore (8.1) is valid in this case.

We next suppose that $C_1 \in \mathfrak{B}^{\infty}_{\mathbf{J}}$. Then $\mathbf{K} \subsetneq \mathbf{J}$. Let ε be an element of $\mathring{P}^{\vee}$ such that $(\varepsilon \,|\, \alpha_i) > 0$ for all $j \in \mathring{\mathbf{I}} \setminus \mathbf{K}$ and $(\varepsilon \,|\, \alpha_k) = 0$ for all $k \in \mathbf{K}$, $\boldsymbol{s}$ an element of $\mathcal{W}_{\mathbf{K}}$ such that $[\boldsymbol{s}] = y$, and $\boldsymbol{s}_2$ an element of $\mathcal{W}^*_{\mathbf{K}}$ such that $\Phi^*_{\mathbf{K}}([\boldsymbol{s}_2]) = B$. Set $\widetilde{C_1} := C_1 \amalg \Delta^{\mathbf{J}}(1,-)$ and $\widetilde{C_2} := C_2 \amalg \Delta^{\mathbf{J}}(1,-)$. Then $\widetilde{C_1} = \nabla(\mathbf{K}, w, y)$ and $\widetilde{C_2} = \widetilde{C_1} \amalg wyB$. Thus, by Proposition 3.8, we have

$$\widetilde{C_1} = \cup_{n\geq 0}\Phi(w\,[\widetilde{\boldsymbol{s}}]\,t^n_{\varepsilon}), \tag{8.4}$$
$$\widetilde{C_2} = \cup_{n\geq 0}\cup_{p=1}^{\ell(\boldsymbol{s}_2)} \Phi(w\,[\widetilde{\boldsymbol{s}}]\,t^n_{\varepsilon}\,[\widetilde{\boldsymbol{s}_2{}_{|p}}]). \tag{8.5}$$

By Proposition 7.6, it follows that

$$A_{\mathbf{J}}(C_1)^c = \cap_{n\geq 0}\{A(w\,[\widetilde{\boldsymbol{s}}]\,t^n_{\varepsilon})^c \cap U^+_{\mathbf{J}}\}, \tag{8.6}$$
$$A_{\mathbf{J}}(C_2) = \cup_{n\geq 0}\cup_{p=1}^{\ell(\boldsymbol{s}_2)} \{A(w\,[\widetilde{\boldsymbol{s}}]\,t^n_{\varepsilon}\,[\widetilde{\boldsymbol{s}_2{}_{|p}}]) \cap U^+_{\mathbf{J}}\}, \tag{8.7}$$

where

$$A(w\,[\widetilde{\boldsymbol{s}}]\,t^n_{\varepsilon})^c \cap U^+_{\mathbf{J}} = \{\, u \in U^+_{\mathbf{J}} \mid T^{-n}_{\varepsilon}\,{}_{\mathbf{J}}T^{-1}_y T^{-1}_w(u) \in U^+ \,\}, \tag{8.8}$$
$$A(w\,[\widetilde{\boldsymbol{s}}]\,t^n_{\varepsilon}\,[\widetilde{\boldsymbol{s}_2{}_{|p}}]) \cap U^+_{\mathbf{J}} = \{\, u \in U^+_{\mathbf{J}} \mid T^{-1}_{[\widetilde{\boldsymbol{s}_2{}_{|p}}]} T^{-n}_{\varepsilon}\,{}_{\mathbf{J}}T^{-1}_y T^{-1}_w(u) \in U^{\leq 0} \,\}. \tag{8.9}$$

Here, by Lemma 5.4(3) we have the claim that $T_\varepsilon^{-n}(x) = x$ for all $x \in A_{\mathbf{K}}(B)$. Combining the claim with (8.6) and (8.8), we get $T_w \cdot {}_{\mathbf{J}}T_y A_{\mathbf{K}}(B) \subset A_{\mathbf{J}}(C_1)^c$. Combining the claim with (8.9) and the equality ${}_{\mathbf{K}}T_{[\boldsymbol{s}_2|_p]} = T_{[\widetilde{\boldsymbol{s}_2|_p}]}|_{U_{\mathbf{K}}}$, we see that $T_w \cdot {}_{\mathbf{J}}T_y A_{\mathbf{K}}([\boldsymbol{s}_2|_p]) \subset A(w\,[\widetilde{\boldsymbol{s}}]\,t_\varepsilon^n\,[\widetilde{\boldsymbol{s}_2|_p}]) \cap U_{\mathbf{J}}^+$ for all $1 \leq p \leq \ell(\boldsymbol{s}_2)$, and hence $T_w \cdot {}_{\mathbf{J}}T_y A_{\mathbf{K}}(B) \subset A_{\mathbf{J}}(C_2)$ by (8.7). Therefore we get $T_w \cdot {}_{\mathbf{J}}T_y A_{\mathbf{K}}(B) \subset A_{\mathbf{J}}(C_1)^c \cap A_{\mathbf{J}}(C_2)$. By (7.20) in Proposition 7.4(1), we see that the multiplication $m\colon A_{\mathbf{J}}(C_1) \otimes \{A_{\mathbf{J}}(C_1)^c \cap A_{\mathbf{J}}(C_2)\} \to A_{\mathbf{J}}(C_2)$ is an isomorphism of vector spaces, which induces the injective linear mapping:

$$\varphi\colon A_{\mathbf{J}}(C_1) \otimes T_w \cdot {}_{\mathbf{J}}T_y A_{\mathbf{K}}(B) \hookrightarrow A_{\mathbf{J}}(C_2). \tag{8.10}$$

Since $C_1 \amalg wyB = C_2$, by Proposition 5.19(1) and (7.15), we see that

$$\dim_{\mathbb{Q}(q)}(\operatorname{Im}\varphi)_\mu = \dim_{\mathbb{Q}(q)} A_{\mathbf{J}}(C_2)_\mu = \sharp\{\,\mathbf{c}\colon C_2 \to \mathbb{Z}_+ \mid \sum_{\beta \in C_2} \mathbf{c}(\beta)\beta = \mu\} \tag{8.11}$$

for each $\mu \in Q_{\mathbf{J}+}$, where $(\operatorname{Im}\varphi)_\mu := (\operatorname{Im}\varphi) \cap U_\mu^+$. This implies that $\varphi = m$ with the equality (8.1).

The (8.2) and (8.3) follow immediately from (8.1), (7.20), and (7.21). □

Corollary 8.2. *Suppose that B is a real biconvex set in $\Delta_{\mathbf{J}+}$ satisfying $B \subset \Delta_{\mathbf{J}}(w,-)$ for some $w \in \mathring{W}_{\mathbf{J}}$ and that*

$$B = \nabla_{\mathbf{J}}(\mathbf{K}, w^{\mathbf{K}}, y), \qquad B \amalg w^{\mathbf{K}} y \Delta_{\mathbf{K}}(\overline{y}^{-1} w_{\mathbf{K}}, -) = \Delta_{\mathbf{J}}(w,-)$$

for some $\mathbf{K} \subset \mathbf{J}$ and $y \in W_{\mathbf{K}}$. Then the multiplication defines the following isomorphisms of $\mathbb{Q}(q)$-vector spaces:

$$A_{\mathbf{J}}(B) \otimes T_{w^{\mathbf{K}}} \cdot {}_{\mathbf{J}}T_y A_{\mathbf{K}}(\overline{y}^{-1} w_{\mathbf{K}}, -) \xrightarrow{\sim} A_{\mathbf{J}}(w,-), \tag{8.12}$$

$$T_{w^{\mathbf{K}}} \cdot {}_{\mathbf{J}}T_y A_{\mathbf{K}}(\overline{y}^{-1} w_{\mathbf{K}}, -) \otimes A_{\mathbf{J}}(B) \xrightarrow{\sim} A_{\mathbf{J}}(w,-), \tag{8.13}$$

$$T_{w^{\mathbf{K}}} \cdot {}_{\mathbf{J}}T_y A_{\mathbf{K}}(\overline{y}^{-1} w_{\mathbf{K}}, -) \otimes A_{\mathbf{J}}(w,-)^c \xrightarrow{\sim} A_{\mathbf{J}}(B)^c, \tag{8.14}$$

$$A_{\mathbf{J}}(w,-)^c \otimes T_{w^{\mathbf{K}}} \cdot {}_{\mathbf{J}}T_y A_{\mathbf{K}}(\overline{y}^{-1} w_{\mathbf{K}}, -) \xrightarrow{\sim} A_{\mathbf{J}}(B)^c. \tag{8.15}$$

Proof. By (8.1), we have

$$A_{\mathbf{J}}(B)^c \cap A_{\mathbf{J}}(w,-) = T_{w^{\mathbf{K}}} \cdot {}_{\mathbf{J}}T_y A_{\mathbf{K}}(\overline{y}^{-1} w_{\mathbf{K}}, -). \tag{8.16}$$

Hence, (8.12) and (8.13) follow from (8.2) and (8.3), respectively. It follows from (7.22) and (7.23) in Proposition 7.4(1) that the multiplication defines the following isomorphisms of vector spaces:

$$\{A_{\mathbf{J}}(B)^c \cap A_{\mathbf{J}}(w,-)\} \otimes A_{\mathbf{J}}(w,-)^c \xrightarrow{\sim} A_{\mathbf{J}}(B)^c, \tag{8.17}$$

$$A_{\mathbf{J}}(w,-)^c \otimes \{A_{\mathbf{J}}(B)^c \cap A_{\mathbf{J}}(w,-)\} \xrightarrow{\sim} A_{\mathbf{J}}(B)^c, \tag{8.18}$$

which imply (8.14) and (8.15). □

Definition 8.3. Let Λ be a totally ordered set with $\preceq$ the total order on Λ. Then we call a subset $I \subset \Lambda$ a *section* of Λ with respect to $\preceq$ if $[\lambda,\mu]_\preceq \subset I$ for all $\lambda, \mu \in I$ satisfying $\lambda \prec \mu$, where $[\lambda,\mu]_\preceq := \{\nu \in \Lambda \,|\, \lambda \preceq \nu \preceq \mu\}$. If, in addition, $I \prec (\Lambda \setminus I)$ then we call I an *initial section* of Λ with respect to $\preceq$. Moreover, for each $\lambda \in \Lambda$ we set

$$(*,\lambda]_\preceq := \{\mu \in \Lambda \,|\, \mu \preceq \lambda\}, \qquad (\lambda,*)_\preceq := \Lambda \setminus (*,\lambda]_\preceq,$$
$$(*,\lambda)_\preceq := \{\mu \in \Lambda \,|\, \mu \prec \lambda\}, \qquad [\lambda,*)_\preceq := \Lambda \setminus (*,\lambda)_\preceq.$$

Let A be an associative algebra with the unit 1 over a commutative ring R, and $\{X_\lambda \mid \lambda \in \Lambda\}$ a subset of A indexed by the totally ordered set Λ. For each finitely supported function $\mathbf{c}\colon \Lambda \to \mathbb{Z}_+$, we set

$$X_{\prec}^{\mathbf{c}} := X_{\lambda_1}^{\mathbf{c}(\lambda_1)} \cdot X_{\lambda_2}^{\mathbf{c}(\lambda_2)} \cdots X_{\lambda_m}^{\mathbf{c}(\lambda_m)}, \quad X_{\succ}^{\mathbf{c}} := X_{\lambda_m}^{\mathbf{c}(\lambda_m)} \cdots X_{\lambda_2}^{\mathbf{c}(\lambda_2)} \cdot X_{\lambda_1}^{\mathbf{c}(\lambda_1)}, \tag{8.19}$$

where $\mathrm{supp}(\mathbf{c}) = \{\lambda_1, \lambda_2, \cdots, \lambda_m\}$ with $\lambda_1 \prec \lambda_2 \prec \cdots \prec \lambda_m$. Here we set $X_{\prec}^{\mathbf{c}} = X_{\succ}^{\mathbf{c}} := 1$ if $\mathrm{supp}(\mathbf{c}) = \emptyset$. We denote by $X_{\prec}(\Lambda)$ (resp. $X_{\succ}(\Lambda)$) the set of all $X_{\prec}^{\mathbf{c}}$ (resp. $X_{\succ}^{\mathbf{c}}$). For each $\Sigma \subset \Lambda$, we set

$$X_{\prec}(\Sigma) := \{\, E_{\prec}^{\mathbf{c}} \mid \mathrm{supp}(\mathbf{c}) \subset \Sigma\}, \quad X_{\succ}(\Sigma) := \{\, E_{\succ}^{\mathbf{c}} \mid \mathrm{supp}(\mathbf{c}) \subset \Sigma\}. \tag{8.20}$$

Then we call $X_{\prec}(\Lambda)$ (resp. $X_{\succ}(\Lambda)$) a *convex basis* of A if $X_{\prec}(I)$ (resp. $X_{\succ}(I)$) is a free R-basis of the R-subalgebra $\langle X_\lambda \mid \lambda \in I \rangle_{R-alg}$ of A for each section I of Λ with respect to $\preceq$.

Theorem 8.4. *Let $(n, \mathbf{K}_\bullet, y_\bullet, \boldsymbol{s}_\bullet)$ be an element of $\mathbb{N}_{\sharp\mathbf{J}} \times \boldsymbol{\mathcal{C}}_n\mathbf{J} \times W_{\mathbf{K}_\bullet} \times \mathcal{W}_{\mathbf{K}_\bullet}^{\infty}$ satisfying the conditions (3.1) and (3.2) (cf. Theorem 3.3(2)), and $\preceq$ the convex order on $\Delta_{\mathbf{J}}(w,-)$ associated with the $(n, \mathbf{K}_\bullet, y_\bullet, \boldsymbol{s}_\bullet)$. For each $\alpha \in \Delta_{\mathbf{J}}(w,-)$, we define a weight vector $E_\alpha = E_{\preceq,\alpha} \in U_{\mathbf{J}}^+$ with weight α by setting*

$$\begin{aligned} E_\alpha = E_{\preceq,\alpha} &:= T_{w^{\mathbf{K}_{i-1}}} \cdot {}_{\mathbf{J}}T_{y_{i-1}}(E_{\boldsymbol{s}_{i-1}}(p)) \\ &= T_{w^{\mathbf{K}_{i-1}}} \cdot {}_{\mathbf{J}}T_{y_{i-1}} \cdot {}_{\mathbf{J}}T_{\boldsymbol{s}_{i-1}(1)} \cdot \cdots \cdot {}_{\mathbf{J}}T_{\boldsymbol{s}_{i-1}(p-1)}(E_{\boldsymbol{s}_{i-1}(p)}), \end{aligned} \tag{8.21}$$

where $\alpha = w^{\mathbf{K}_{i-1}} y_{i-1} \phi_{\boldsymbol{s}_{i-1}}(p)$ with $i \in \mathbb{N}_n$ and $p \in \mathbb{N}$. Then the sets $E_{\prec}(\Delta_{\mathbf{J}}(w,-))$ and $E_{\succ}(\Delta_{\mathbf{J}}(w,-))$, respectively, are convex bases of the $\mathbb{Q}(q)$-algebra $A_{\mathbf{J}}(w,-)$ and of the $\mathcal{A}_1$-algebra ${}_{\mathcal{A}_1}A_{\mathbf{J}}(w,-) := A_{\mathbf{J}}(w,-) \cap {}_{\mathcal{A}_1}U^+$.

Moreover, if I is an initial section with respect to $\preceq$, then I is a real biconvex set in $\Delta_{\mathbf{J}+}$ and the following equalities hold:

$$\langle\, E_\alpha \mid \alpha \in I \,\rangle_{\mathbb{Q}(q)-alg} = A_{\mathbf{J}}(I), \tag{8.22}$$

$$\langle\, E_\alpha \mid \alpha \in I^c \,\rangle_{\mathbb{Q}(q)-alg} = A_{\mathbf{J}}(I)^c \cap A_{\mathbf{J}}(w,-), \tag{8.23}$$

where $I^c := \Delta_{\mathbf{J}}(w,-) \setminus I$.

Proof. By (3.1)(3.2) in Theorem 3.3(2), we have

$$\Delta(w,-) = \amalg_{i=1}^{n} w^{\mathbf{K}_{i-1}} y_{i-1} \Phi_{\mathbf{K}_{i-1}}^{\infty}([\boldsymbol{s}_{i-1}]), \tag{8.24}$$

$$C_{i-1} \amalg w^{\mathbf{K}_{i-1}} y_{i-1} \Phi_{\mathbf{K}_{i-1}}^{\infty}([\boldsymbol{s}_{i-1}]) = C_i \in \mathfrak{B}_{\mathbf{J}}^{\infty} \quad \text{for each } 1 \leqq i \leqq n, \tag{8.25}$$

where $y_0 := 1$ and $C_0 := \emptyset$. We set $B_{i-1} := \Phi_{\mathbf{K}_{i-1}}^{\infty}([\boldsymbol{s}_{i-1}])$ for each $i \in \mathbb{N}_n$. Then, by (8.2) and (8.25), we see that the multiplication defines the following $\mathbb{Q}(q)$-linear isomorphism:

$$A_{\mathbf{J}}(C_{i-1}) \otimes T_{w^{\mathbf{K}_{i-1}}} \cdot {}_{\mathbf{J}}T_{y_{i-1}} A_{\mathbf{K}_{i-1}}(B_{i-1}) \xrightarrow{\sim} A_{\mathbf{J}}(C_i) \tag{8.26}$$

for each $i \in \mathbb{N}_n$. Since $C_n = \Delta_{\mathbf{J}}(w,-)$ and $C_1 = B_0$, the multiplication defines the following $\mathbb{Q}(q)$-linear isomorphisms:

$$\otimes_{j=1}^{n} T_{w^{\mathbf{K}_{j-1}}} \cdot {}_{\mathbf{J}}T_{y_{j-1}} A_{\mathbf{K}_{j-1}}(B_{j-1}) \xrightarrow{\sim} A_{\mathbf{J}}(w,-), \tag{8.27}$$

$$\otimes_{j=1}^{i} T_{w^{\mathbf{K}_{j-1}}} \cdot {}_{\mathbf{J}}T_{y_{j-1}} A_{\mathbf{K}_{j-1}}(B_{j-1}) \xrightarrow{\sim} A_{\mathbf{J}}(C_i), \tag{8.28}$$

$$\otimes_{j=i+1}^{n} T_{w^{\mathbf{K}_{j-1}}} \cdot {}_{\mathbf{J}}T_{y_{j-1}} A_{\mathbf{K}_{j-1}}(B_{j-1}) \xrightarrow{\sim} A_{\mathbf{J}}(C_i)^c \cap A_{\mathbf{J}}(w,-). \tag{8.29}$$

Here, $A_{j-1} := T_{w^{\mathbf{K}_{j-1}}} \cdot {}_{\mathbf{J}}T_{y_{j-1}} A_{\mathbf{K}_{j-1}}(B_{j-1})$ is located on the left side of $A_{j'-1}$ in the tensor products above if $j < j'$. By (8.27), we see that $E_{\prec}(\Delta_{\mathbf{J}}(w,-))$ is a basis of $A_{\mathbf{J}}(w,-)$. Moreover, by Lemma 4.5(1) we see that $E_{\prec}(\Delta_{\mathbf{J}}(w,-))$ is a subset of

${}_{\mathcal{A}_1}A_{\mathbf{J}}(w,-)\setminus(q-1){}_{\mathcal{A}_1}A_{\mathbf{J}}(w,-)$, and hence that the set $E_{\prec}(\Delta_{\mathbf{J}}(w,-))$ is also a basis of ${}_{\mathcal{A}_1}A_{\mathbf{J}}(w,-)$ over $\mathcal{A}_1$ by Proposition 4.2 and Lemma 4.3.

We next prove (8.22)(8.23). Since I is an initial section, it is easy to see that I is a real biconvex set in $\Delta_{\mathbf{J}+}$. Let us consider in the case where $I=(*,\alpha]_{\preceq}$, and let $i\in\mathbb{N}_n$ and $p\in\mathbb{N}$ be unique elements such that $\alpha=w^{\mathbf{K}_{i-1}}y_{i-1}\phi_{\boldsymbol{s}_{i-1}}(p)$. We put $x:=[\boldsymbol{s}_{i-1}|_p]$ and $B'_{i-1}:=x^{-1}\{B_{i-1}\setminus\Phi_{\mathbf{K}_{i-1}}(x)\}$. Then we see that

$$I=C_{i-1}\amalg w^{\mathbf{K}_{i-1}}y_{i-1}\Phi_{\mathbf{K}_{i-1}}(x), \tag{8.30}$$

$$C_i=I\amalg w^{\mathbf{K}_{i-1}}y_{i-1}xB'_{i-1}. \tag{8.31}$$

By (8.2) and (8.30), we see that the multiplication defines the following $\mathbb{Q}(q)$-linear isomorphism:

$$A_{\mathbf{J}}(C_{i-1})\otimes T_{w^{\mathbf{K}_{i-1}}}\cdot{}_{\mathbf{J}}T_{y_{i-1}}A_{\mathbf{K}_{i-1}}(x)\xrightarrow{\sim}A_{\mathbf{J}}(I). \tag{8.32}$$

By (8.28) with replacing i by $i-1$ and (8.32), we see that $E_{\prec}(I)$ is a basis of $A_{\mathbf{J}}(I)$ and that (8.22) holds for $I=(*,\alpha]_{\preceq}$. By (8.1) and (8.31), we have

$$A_{\mathbf{J}}(I)^c\cap A_{\mathbf{J}}(C_i)=T_{w^{\mathbf{K}_{i-1}}}\cdot{}_{\mathbf{J}}T_{y_{i-1}}\cdot{}_{\mathbf{J}}T_xA_{\mathbf{K}_{i-1}}(B'_{i-1}). \tag{8.33}$$

Since $I\subset C_i\subset\Delta_{\mathbf{J}}(w,-)$, it follows from (7.22) that the multiplication defines the following $\mathbb{Q}(q)$-linear isomorphism:

$$\{A_{\mathbf{J}}(I)^c\cap A_{\mathbf{J}}(C_i)\}\otimes\{A_{\mathbf{J}}(C_i)^c\cap A_{\mathbf{J}}(w,-)\}\xrightarrow{\sim}A_{\mathbf{J}}(I)^c\cap A_{\mathbf{J}}(w,-). \tag{8.34}$$

By (8.29)(8.33)(8.34), we see that $E_{\prec}(I^c)$ is a basis of $A_{\mathbf{J}}(I)^c\cap A_{\mathbf{J}}(w,-)$ and that (8.23) holds for $I=(*,\alpha]_{\preceq}$. Similarly, we can prove the assertions in the case where $I=(*,\alpha)_{\preceq}$.

For each $\alpha\preceq\beta$, we see that $E_{\prec}([\alpha,*)_{\preceq})\cap E_{\prec}((*,\beta]_{\preceq})=E_{\prec}([\alpha,\beta]_{\preceq})$ and

$$\langle E_\eta\,|\,\eta\in[\alpha,*)_{\preceq}\rangle_{\mathbb{Q}(q)-alg}\cap\langle E_\eta\,|\,\eta\in(*,\beta]_{\preceq}\rangle_{\mathbb{Q}(q)-alg}=\langle E_\eta\,|\,\eta\in[\alpha,\beta]_{\preceq}\rangle_{\mathbb{Q}(q)-alg}.$$

Thus $E_{\prec}(I)$ is a basis of $\langle E_\eta\,|\,\eta\in I\rangle_{\mathbb{Q}(q)-alg}$ for $I=[\alpha,\beta]_{\preceq}$. Similarly, we can prove that $E_{\prec}(I)$ is a basis of $\langle E_\eta\,|\,\eta\in I\rangle_{\mathbb{Q}(q)-alg}$ for each section I with respect to $\preceq$. Therefore, the set $E_{\prec}(\Delta_{\mathbf{J}}(w,-))$ is a convex basis of $A_{\mathbf{J}}(w,-)$, and hence the set is a convex basis of ${}_{\mathcal{A}_1}A_{\mathbf{J}}(w,-)$ over $\mathcal{A}_1$ by Proposition 4.2 and Lemma 4.3. The proof of the assertion for $E_{\succ}(\Delta_{\mathbf{J}}(w,-))$ is quite similar. □

Proposition 8.5. (1) *The following equalities hold*:

$$A_{\mathbf{J}}(1,+)=\langle\,x^+_{j,m}\mid j\in\mathbf{J},\,m\in\mathbb{Z}_+\,\rangle_{\mathbb{Q}(q)-alg}, \tag{8.35}$$

$$A_{\mathbf{J}}(1,-)=\langle\,E_{\delta-\varepsilon}\,,\,x^-_{j,n}\mid\varepsilon\in\mathring{\Delta}_{\mathbf{J}+},\,j\in\mathbf{J},\,n\in\mathbb{N}\,\rangle_{\mathbb{Q}(q)-alg}, \tag{8.36}$$

where both $x^+_{j,m}$ and $x^-_{j,n}$ are introduced in Definition 6.1, and $E_{\delta-\varepsilon}$ is introduced in Definition 5.2.

(2) *For each $w\in\mathring{W}$, the following inclusions hold*:

$$[A_{\mathbf{J}}(w,\pm)\,,\,A_{\mathbf{J}}(w,0)]\subset A_{\mathbf{J}}(w,\pm). \tag{8.37}$$

Proof. (1) Since the proof of (8.36) is similar to that of (8.35), we only prove (8.35). Set $X^+_{\mathbf{J}}:=\langle x^+_{j,m}\,|\,j\in\mathbf{J},\,m\in\mathbb{Z}_+\rangle_{\mathbb{Q}(q)-alg}$. Then, by Lemma 6.2(2) we have $A_{\mathbf{J}}(1,+)\supset X^+_{\mathbf{J}}$. To prove the opposite inclusion $A_{\mathbf{J}}(1,+)\subset X^+_{\mathbf{J}}$, let λ be an element of $\mathring{Q}^\vee_{\mathbf{J}}\setminus\{0\}$ such that $\lambda=\sum_{j\in\mathbf{J}}k_j\varepsilon_j$ with $k_j\in\mathbb{N}$ for all $j\in\mathbf{J}$, and $s_1,s_2,\ldots,s_n$ elements of $S_{\mathbf{J}}$ such that $s_1s_2\cdots s_n=t_{-\lambda}$ with $n=\ell_{\mathbf{J}}(t_{-\lambda})$. Here, we define an infinite sequence $\boldsymbol{s}=(\boldsymbol{s}(p))_{p\in\mathbb{N}}\in S^{\mathbb{N}}_{\mathbf{J}}$ by setting $\boldsymbol{s}(p):=s_{\overline{p}}$ for each $p\in\mathbb{N}$, where $\overline{p}\in\mathbb{N}_n$ such that $\overline{p}\equiv p\mod n$. Then the $\boldsymbol{s}$ is an element of $\mathcal{W}^\infty_{\mathbf{J}}$ such that $\Phi^{\infty}_{\mathbf{J}}([\boldsymbol{s}]) = \Delta_{\mathbf{J}}(w,+)$, and hence the convex order $\preceq$ on $\Delta_{\mathbf{J}}(w,+)$ associated with $\boldsymbol{s}$ is of 1-row type (see Theorem 3.3 and Remark 3.4). Since $A_{\mathbf{J}}(1,+) = \langle \Psi E_{\preceq,\alpha} \,|\, \alpha \in \Delta_{\mathbf{J}}(1,+)\rangle_{\mathbb{Q}(q)-alg}$, it suffices to show that $\Psi E_{\preceq,\alpha} \in X^+_{\mathbf{J}}$ for all $\alpha \in \Delta_{\mathbf{J}}(1,+)$. We use the induction on $\mathrm{ht}(\overline{\alpha})$. Firstly, we consider the case where $\mathrm{ht}(\overline{\alpha}) = 1$. Then $\alpha = m\delta + \alpha_j$ with $(j,m) \in \mathbf{J} \times \mathbb{Z}_+$. Hence, by Lemma 6.2(3) we see that $\Psi E_{\preceq, m\delta+\alpha_j} = x^+_{j,m} \in X^+_{\mathbf{J}}$. Secondly, we consider the case where $\mathrm{ht}(\overline{\alpha}) \geq 2$. Let $[\beta,\gamma]_{\preceq}$ be a minimal section of $\Delta_{\mathbf{J}}(1,+)$ satisfying $\alpha = \beta + \gamma$. By Theorem 8.4 and Proposition 4.2, we see that there exist elements $c_1, c_2 \in \mathcal{A}_1 \setminus (q-1)\mathcal{A}_1$ such that $E_{\preceq,\gamma}E_{\preceq,\beta} = c_1 E_{\preceq,\alpha} + c_2 E_{\preceq,\beta}E_{\preceq,\gamma}$. Since $\mathrm{ht}(\overline{\beta}), \mathrm{ht}(\overline{\gamma}) < \mathrm{ht}(\overline{\alpha})$, by the hypothesis of the induction, we see that

$$\Psi(E_{\preceq,\alpha}) = \frac{1}{c_1}\Psi(E_{\preceq,\beta})\Phi(E_{\preceq,\gamma}) - \frac{c_2}{c_1}\Psi(E_{\preceq,\gamma})\Psi(E_{\preceq,\gamma}) \in X^+_{\mathbf{J}}.$$

(2) By Proposition 6.5 and (8.35), we have

$$[A_{\mathbf{J}}(1,+)\,,\, A_{\mathbf{J}}(1,0)] \subset A_{\mathbf{J}}(1,+). \tag{8.38}$$

Since $\Delta_{\mathbf{J}}(1,+) = \Delta_{\mathbf{J}}(w_\circ,-)$, by (5.37)(5.38)(5.46), we have

$$[A_{\mathbf{J}}(w_\circ,-)\,,\, A_{\mathbf{J}}(w_\circ,0)] \subset A_{\mathbf{J}}(w_\circ,-), \tag{8.39}$$

where $w_\circ$ is the longest element of $\mathring{W}_{\mathbf{J}}$. Set $w' = w_\circ w^{-1}$. Then we have $T_{w_\circ} = T_w T_{w'}$, and hence $T_{w'}A_{\mathbf{J}}(w,0) = A_{\mathbf{J}}(w_\circ,0)$. Since the multiplication defines the $\mathbb{Q}(q)$-linear isomorphism $U_<(w') \otimes T_{w'}U_{\mathbf{J},<}(w,-) \to U_{\mathbf{J},<}(w_\circ,-)$, we have $T_{w'}A_{\mathbf{J}}(w,-) \subset A_{\mathbf{J}}(w_\circ,-)$. Therefore, by (8.39), we have

$$[T_{w'}A_{\mathbf{J}}(w,-)\,,\, T_{w'}A_{\mathbf{J}}(w,0)] \subset A_{\mathbf{J}}(w_\circ,-).$$

Since $[T_{w'}A_{\mathbf{J}}(w,-)\,,\, T_{w'}A_{\mathbf{J}}(w,0)] \in A_{\mathbf{J}}(w_\circ,-) \cap A(w')^c$, we have

$$[T_{w'}A_{\mathbf{J}}(w,-)\,,\, T_{w'}A_{\mathbf{J}}(w,0)] \subset T_{w'}A_{\mathbf{J}}(w,-),$$

and hence $[A_{\mathbf{J}}(w,-), A_{\mathbf{J}}(w,0)] \subset A_{\mathbf{J}}(w,-)$. By (5.37)(5.38)(5.46), we have

$$[A_{\mathbf{J}}(w,+)\,,\, \Psi A_{\mathbf{J}}(ww_\circ,0)] \subset A_{\mathbf{J}}(w,+),$$

and hence $[A_{\mathbf{J}}(w,+)\,,\, A_{\mathbf{J}}(w,0)] \subset A_{\mathbf{J}}(w,+)$. □

Theorem 8.6. *Let $\preceq$ be an arbitrary convex order on $\Delta_{\mathbf{J}+}$, and $w \in \mathring{W}_{\mathbf{J}}$ the unique element such that*

$$\Delta_{\mathbf{J}}(w,-) \prec \Delta^{im}_{+} \prec \Delta_{\mathbf{J}}(w,+). \tag{8.40}$$

We define $\preceq_-$, $\preceq_0$, and $\preceq_+$ to be the restriction of $\preceq$ to $\Delta_{\mathbf{J}}(w,-)$, Δ^{im}_+, and $\Delta_{\mathbf{J}}(w,+)$, respectively, and define a total order $\tilde{\preceq}_0$ on the following set

$$\tilde{\Delta}^{im}_{\mathbf{J}+} = \Delta^{im}_{+} \times \mathbf{J} = \{\, (n\delta, j) \mid n \in \mathbb{N},\, j \in \mathbf{J}\,\}$$

by setting

$$(n\delta, j) \;\tilde{\prec}_0\; (n'\delta, j') \Longleftrightarrow \begin{cases} n\delta \prec_0 n'\delta & \text{if } n \neq n', \\ j < j' & \text{if } n = n'. \end{cases} \tag{8.41}$$

In addition, we define a total order $\tilde{\preceq}$ on the following set

$$\tilde{\Delta}_{\mathbf{J}+} = \Delta^{re}_{\mathbf{J}+} \amalg \tilde{\Delta}^{im}_{\mathbf{J}+} = \Delta_{\mathbf{J}}(w,-) \amalg \tilde{\Delta}^{im}_{\mathbf{J}+} \amalg \Delta_{\mathbf{J}}(w,+)$$

by extending $\preceq_-$, $\tilde{\preceq}_0$, and $\preceq_+$ such as

$$\Delta_{\mathbf{J}}(w,-) \;\tilde{\prec}\; \tilde{\Delta}^{im}_{\mathbf{J}+} \;\tilde{\prec}\; \Delta_{\mathbf{J}}(w,+). \tag{8.42}$$

For each $\eta \in \tilde{\Delta}_{\mathbf{J}+}$*, we set*

$$E_\eta = E_{\preceq,\eta} := \begin{cases} E_{\preceq_-,\eta} & \text{if } \eta \in \Delta_{\mathbf{J}}(w,-), \\ T_w(I_{j,n}) & \text{if } \eta = (n\delta, j) \in \tilde{\Delta}^{im}_{\mathbf{J}+}, \\ \Psi(E_{\preceq^{op}_+,\eta}) & \text{if } \eta \in \Delta_{\mathbf{J}}(w,+), \end{cases} \tag{8.43}$$

where $\preceq^{op}_+$ *is the opposite order of* $\preceq_+$*. Then the sets* $E_\prec(\tilde{\Delta}_{\mathbf{J}+})$ *and* $E_\succ(\tilde{\Delta}_{\mathbf{J}+})$*, respectively, are convex bases of the* $\mathbb{Q}(q)$*-algebra* $U^+_{\mathbf{J}}$ *and of the* $\mathcal{A}_1$*-algebra* ${}_{\mathcal{A}_1}U^+_{\mathbf{J}} := {}_{\mathcal{A}_1}U^+ \cap U^+_{\mathbf{J}}$*.*

Moreover, for each $\eta, \zeta \in \tilde{\Delta}_{\mathbf{J}+}$ *satisfying* $\eta \,\tilde{\prec}\, \zeta$*, the following equalities hold*:

$$[E_\eta\,,\,E_\zeta]_q = \sum_{\mathrm{supp}(\mathbf{c}) \subset (\eta,\zeta)_{\tilde{\preceq}}} h_{\mathbf{c}} E^{\mathbf{c}}_{\prec}, \tag{8.44}$$

$$[E_\eta\,,\,E_\zeta]_q = \sum_{\mathrm{supp}(\mathbf{c}) \subset (\eta,\zeta)_{\tilde{\preceq}}} g_{\mathbf{c}} E^{\mathbf{c}}_{\succ}, \tag{8.45}$$

where $h_{\mathbf{c}}, g_{\mathbf{c}} \in \mathcal{A}_1$*.*

Proof. By Proposition 6.8, Proposition 7.1(1), Theorem 8.4, and Proposition 7.2(2), we see that $E_\prec(\tilde{\Delta}_{\mathbf{J}+})$ is a basis of $U^+_{\mathbf{J}}$. Moreover, by Lemma 4.5(1) and Lemma 6.6(1), we see that $E_\prec(\tilde{\Delta}_{\mathbf{J}+})$ is a subset of ${}_{\mathcal{A}_1}U^+_{\mathbf{J}} \setminus (q-1)_{\mathcal{A}_1}U^+_{\mathbf{J}}$. Hence, it follows from Proposition 4.2 and Lemma 4.3 that the set $E_\prec(\tilde{\Delta}_{\mathbf{J}+})$ is also a basis of the $\mathcal{A}_1$-algebra ${}_{\mathcal{A}_1}U^+_{\mathbf{J}}$.

Suppose that $\eta \in \Delta_{\mathbf{J}}(w,-)$. Since $(*,\eta]_{\tilde{\preceq}} = (*,\eta]_{\preceq_-}$, by (8.22), we see that $E_\prec((*,\eta]_{\tilde{\preceq}})$ is a basis of $A_{\mathbf{J}}((*,\eta]_{\tilde{\preceq}})$. By (8.23), (7.17), and (7.22), we see that $E_\prec((\eta,*)_{\tilde{\preceq}})$ is a basis of $A_{\mathbf{J}}((*,\eta]_{\tilde{\preceq}})^c$.

We next suppose that $\eta \in \Delta_{\mathbf{J}}(w,+)$. Here we remark that Ψ is an anti-automorphism of the $\mathbb{Q}(q)$-algebra $U^+_{\mathbf{J}}$. Since $(\eta,*)_{\tilde{\preceq}} = (\eta,*)_{\preceq_+}$, by (8.22), we see that $E_\prec((\eta,*)_{\tilde{\preceq}})$ is a basis of $\Psi A_{\mathbf{J}}((\eta,*)_{\tilde{\preceq}})$. By (8.23), (7.18), and (7.22), we see that $E_\prec((*,\eta]_{\tilde{\preceq}})$ is a basis of $\Psi A_{\mathbf{J}}((\eta,*)_{\tilde{\preceq}})^c$.

We next suppose that $\eta \in \tilde{\Delta}^{im}_{\mathbf{J}+}$. By Proposition 6.8, (7.3), (7.16), Theorem 8.4, and Proposition 8.5(2), we see that $E_\prec(I)$ is a basis of the subalgebra $\langle E_\eta \,|\, \eta \in I\,\rangle_{\mathbb{Q}(q)-alg}$ in the case where $I = (*,\eta]_{\tilde{\preceq}}$ or $I = (\eta,*)_{\tilde{\preceq}}$.

Therefore we see that $E_\prec(I)$ is a basis of $\langle E_\eta \,|\, \eta \in I\,\rangle_{\mathbb{Q}(q)-alg}$ in the cases where $I = (*,\eta]_{\tilde{\preceq}}$ or $I = (\eta,*)_{\tilde{\preceq}}$ for each $\eta \in \Delta_{\mathbf{J}+}$. Similarly, we can prove that $E_\prec(I)$ is a basis of $\langle E_\eta \,|\, \eta \in I\,\rangle_{\mathbb{Q}(q)-alg}$ in the case where $I = (*,\eta)_{\tilde{\preceq}}$ or $I = [\eta,*)_{\tilde{\preceq}}$ for each $\eta \in \Delta_{\mathbf{J}+}$.

For each $\eta \tilde{\preceq} \zeta$, we see that $E_\prec([\eta,*)_{\tilde{\preceq}}) \cap E_\prec((*,\zeta]_{\tilde{\preceq}}) = E_\prec([\eta,\zeta]_{\tilde{\preceq}})$ and

$$\langle E_\eta \,|\, \eta \in [\eta,*)_{\tilde{\preceq}}\,\rangle_{\mathbb{Q}(q)-alg} \cap \langle E_\eta \,|\, \eta \in (*,\zeta]_{\tilde{\preceq}}\,\rangle_{\mathbb{Q}(q)-alg} = \langle E_\eta \,|\, \eta \in [\eta,\zeta]_{\tilde{\preceq}}\,\rangle_{\mathbb{Q}(q)-alg}.$$

Thus we see that $E_\prec(I)$ is a basis of $\langle E_\eta \,|\, \eta \in I\,\rangle_{\mathbb{Q}(q)-alg}$ in the case where $I = [\eta,\zeta]_{\tilde{\preceq}}$. Similarly, we can prove that $E_\prec(I)$ is a basis of $\langle E_\eta \,|\, \eta \in I\,\rangle_{\mathbb{Q}(q)-alg}$ for each section I with respect to $\preceq$. Therefore $E_\prec(\tilde{\Delta}_{\mathbf{J}+})$ is a convex basis of $U^+_{\mathbf{J}}$. Since $E_\prec(\tilde{\Delta}_{\mathbf{J}+})$ is also a basis of the $\mathcal{A}_1$-algebra ${}_{\mathcal{A}_1}U^+_{\mathbf{J}}$, it is easy to see that $E_\prec(\tilde{\Delta}_{\mathbf{J}+})$ is a convex basis of ${}_{\mathcal{A}_1}U^+_{\mathbf{J}}$.

We next prove (8.44). In the case where $\eta, \zeta \in \Delta_{\mathbf{J}}(w,-)$, we see that

$$E_\zeta E_\eta - a E_\eta E_\zeta = \sum_{\mathrm{supp}(\mathbf{c}) \subset (\eta,\zeta)_{\preceq}} h'_{\mathbf{c}} E^{\mathbf{c}}_{\prec}, \tag{8.46}$$

where $a, h'_{\mathbf{c}} \in \mathcal{A}_1$. Hence it suffices to show that $a = q^{-(\dot{\eta}|\dot{\zeta})}$, where $\dot{\xi} := \mathrm{wt}(E_\xi)$ for each $\xi \in \tilde{\Delta}_{\mathbf{J}+}$. Recall that $\eta = w^{\mathbf{K}_{i-1}} y_{i-1} \phi_{\boldsymbol{s}_{i-1}}(p)$ for some $i \in \mathbb{N}_n$ and $p \in \mathbb{N}$. We put $x := [\boldsymbol{s}_{i-1}|_p]$. Then, by Theorem 8.4, we see that

$$E_\eta \in T_{w_{\mathbf{K}_{i-1}}} \cdot {}_{\mathbf{J}}T_{y_{i-1}} A_{\mathbf{K}_{i-1}}(x), \qquad E_\prec((\eta,\zeta]_\preceq) \subset T_{w_{\mathbf{K}_{i-1}}} \cdot {}_{\mathbf{J}}T_{y_{i-1}} A_{\mathbf{K}_{i-1}}(x)^c,$$

where $(\eta,\zeta]_\preceq := (\eta,*)_\preceq \cap (*,\zeta]_\preceq$. By applying $T = {}_{\mathbf{J}}T_x^{-1} {}_{\mathbf{J}}T_{y_{i-1}}^{-1} T_{w_{\mathbf{K}_{i-1}}}^{-1}$ to the both sides of (8.46), we see that

$$\begin{aligned}
&(\text{the left hand side}) = -T(E_\zeta) K_{\boldsymbol{s}_{i-1}(p)}^{-1} F_{\boldsymbol{s}_{i-1}(p)} + a K_{\boldsymbol{s}_{i-1}(p)}^{-1} F_{\boldsymbol{s}_{i-1}(p)} T(E_\zeta) \\
&\quad = (a - q^{-(\dot{\eta}|\dot{\zeta})}) K_{\boldsymbol{s}_{i-1}(p)}^{-1} F_{\boldsymbol{s}_{i-1}(p)} T(E_\zeta) - q^{-(\dot{\eta}|\dot{\zeta})} K_{\boldsymbol{s}_{i-1}(p)}^{-1} [T(E_\zeta), F_{\boldsymbol{s}_{i-1}(p)}], \quad (8.47) \\
&(\text{the right hand side}) = \sum_{\mathrm{supp}(\mathbf{c}) \subset (\eta,\zeta)_\preceq} h'_{\mathbf{c}} T(E_\prec^{\mathbf{c}}) \in U_{\mathbf{J}}^+. \quad (8.48)
\end{aligned}$$

Since both $T(E_\zeta)$ and ${}_{\mathbf{J}}T_{\boldsymbol{s}_{i-1}(p)} T(E_\zeta)$ are elements of $U_{\mathbf{J}}^+$, we see that

$$K_{\boldsymbol{s}_{i-1}(p)}^{-1} [T(E_\zeta), F_{\boldsymbol{s}_{i-1}(p)}] \in U_{\mathbf{J}}^+ \quad (8.49)$$

by Proposition 3.1.6(a) in [14], Proposition 7.1(3), Theorem 5.12, and Proposition 5.10. By (5.15) and (8.47)(8.48)(8.49), we get $a = q^{-(\dot{\eta}|\dot{\zeta})}$. Similarly, we can prove (8.44) in the case where $\eta, \zeta \in \Delta_{\mathbf{J}}(w,+)$ or the case where $\eta \in \Delta_{\mathbf{J}}(w,-)$ and $\zeta \in \Delta_{\mathbf{J}}(w,+)$. In the case where $\eta, \zeta \in \tilde{\Delta}_{\mathbf{J}+}^{im}$, the (8.44) follows from Proposition 6.8. The proof of (8.45) is quite similar. □

9. Dual convex bases of U^+ and U^- with respect to the q-Killing form

Throughout this section, we assume that $\mathfrak{g}$ is the affine Kac-Moody Lie algebra of the type $X_r^{(1)}$ ($X = A, B, C, D, E, F, G$). Firstly, we introduce a well-known standard $\mathbb{Q}(q)$-bilinear form between $U^{\geq 0}$ and $U^{\leq 0}$, which is called the q-Killing form since it can be regarded as a q-analogue of the Killing form on $\mathfrak{g}$. Secondly, we introduce a Damiani's work concerning detailed computation of values of the q-Killing form on the subalgebras generated by the imaginary root vectors. Thirdly, we will construct the dual convex bases of U^+ and U^- with respect to the q-Killing form. Finally, we will present the multiplicative formula for the R-matrix of $U_q(\mathfrak{g})$ associated with an arbitrary convex order on Δ_+.

Theorem 9.1 ([15])**.** *There exists a unique non-degenerate $\mathbb{Q}(q)$-bilinear form $(\ |\ ) : U^{\geq 0} \times U^{\leq 0} \to \mathbb{Q}(q)$ which satisfies the following equalities*:

$$\begin{aligned}
&(x \,|\, y_1 y_2) = (\Delta(x) \,|\, y_1 \otimes y_2), \qquad (x_1 x_2 \,|\, y) = (x_2 \otimes x_1 \,|\, \Delta(y)), \\
&(K_\mu \,|\, K_\nu) = q^{-(\mu|\nu)}, \qquad (E_i \,|\, K_\nu) = (K_\mu \,|\, F_i) = 0, \qquad (E_i \,|\, F_j) = \delta_{ij}/(q_i^{-1} - q_i),
\end{aligned}$$

where $x, x_1, x_2 \in U^{\geq 0}$, $y, y_1, y_2 \in U^{\leq 0}$, $i, j \in \mathbf{I}$, $\mu, \nu \in P$, and Δ is the coproduct of U defined by

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \Delta(K_\mu) = K_\mu \otimes K_\mu.$$

Here we use the notation $(\ |\ )$ also for the $\mathbb{Q}(q)$-bilinear form $(\ |\ ): (U^{\geq 0})^{\otimes 2} \times (U^{\leq 0})^{\otimes 2} \to \mathbb{Q}(q)$ induced by

$$(x_1 \otimes x_2 \,|\, y_1 \otimes y_2) := (x_1 \,|\, y_1)(x_2 \,|\, y_2).$$

Lemma 9.2 ([10])**.** (1) *For each* $\mu, \nu \in Q_+$, $x \in U^+_\mu$, $y \in U^-_{-\nu}$, *and* $\xi, \eta \in P$, *the following equality holds*:

$$(xK_\xi \mid yK_\eta) = \delta_{\mu\nu} q^{-(\xi|\eta)}(x \mid y). \tag{9.1}$$

Moreover, the restriction of the form $(\ \mid\ )$ *to* $U^+_\mu \times U^-_{-\mu}$ *is non-degenerate.*

(2) *For each* $x \in U^+$ *and* $y \in U^-$, *the following equality holds*:

$$(\Psi(x) \mid \Psi(y)) = (x \mid y).$$

(3) *For each* $i \in \mathbf{I}$, $x \in A(s_i)^c$, $y \in A^-(s_i)^c$, *and* $m, n \in \mathbb{Z}_{\geq 0}$, *the following equality holds*:

$$(xE_i^m \mid yF^n) = \delta_{mn}(x \mid y)(E_i^m \mid F_i^m) \tag{9.2}$$

with the following equality:

$$(E_i^m \mid F_i^m) = (m)_{q_i}!/(q_i^{-1} - q_i)^m. \tag{9.3}$$

Proof. Although the claims are proved in [10] in the case where $\mathfrak{g}$ is an arbitrary finite dimensional simple Lie algebra, the proof can be applied to the untwisted affine case. □

Proposition 9.3. *For each* $y \in W$, $a \in \Psi(A(y)^c)$, *and* $b \in \Psi(A^-(y)^c)$, *the following equality holds*:

$$(T_y(a) \mid T_y(b)) = (a \mid b). \tag{9.4}$$

Proof. Since the claim is clear in the case where $y = 1$, we may assume that $\ell(y) > 1$. We use the induction on $l = \ell(y)$. In the case where $l = 1$, we can apply the proof of Proposition 8.28 in [10] to the case. In the case where $l \geq 2$, there exist $i \in \mathbf{I}$ and $y' \in W$ such that $y = y's_i$ and $\ell(y') = l - 1$. Then we see that $T_y = T_{y'}T_i$, $T_i(a) \in \Psi(A(y')^c)$, and $T_i(b) \in \Psi(A^-(y')^c)$. Hence, by the inductive assumption, we have the following equalities:

$$(T_y(a) \mid T_y(b)) = (T_{y'}T_i(a) \mid T_{y'}T_i(b)) = (T_i(a) \mid T_i(b)) = (a \mid b).$$

□

Proposition 9.4. *Let* B *be an arbitrary element of* $\mathfrak{B}^*$. *If* $x_1 \in A(B)^c$, $x_2 \in A(B)$, $y_1 \in A^-(B)^c$, $y_2 \in A^-(B)$, *then the following equality holds*:

$$(x_1x_2 \mid y_1y_2) = (x_1 \mid y_1)(x_2 \mid y_2). \tag{9.5}$$

Proof. We first prove (9.5) in the case where $B = \Phi(y)$ with $y \in W$. By Proposition 7.2(2), we have $A(B) = A(y) = U_>(y)$ and $A^-(B) = A^-(y) = U^-_>(y)$. We use the induction on $l = \ell(y)$. In the case where $l = 0$, since $y = 1$, we have $U_>(y) = U^-_>(y) = \mathbb{Q}(q)$, which implies (9.5). In the case where $l > 0$, there exist $i \in \mathbf{I}$ and $y' \in W$ such that $y = s_iy'$ and $\ell(y') = l - 1$. By Proposition 5.20(2), there exist $m, n \in \mathbb{Z}_{\geq 0}$, $a_2' \in T_iU_>(y')$, and $b_2' \in T_iU^-_>(y')$ such that $a_2 = a_2'E_i^m$ and $b_2 = b_2'F_i^n$. Then we have $T_i^{-1}(a_2') \in U_>(y')$ and $T_i^{-1}(b_2') \in U^-_>(y')$. In addition, we have $T_i^{-1}(a_1) \in A(y')^c$ and $T_i^{-1}(b_1) \in A^-(y')^c$. By Lemma 9.2(3), Proposition 9.3, and the inductive assumption, we see that

$$\begin{aligned}
(a_1a_2 \mid b_1b_2) &= (a_1a_2'E_i^m \mid b_1b_2'F_i^n) = (a_1a_2' \mid b_1b_2')(E_i^m \mid F_i^n) \\
&= (T_i^{-1}(a_1)T_i^{-1}(a_2') \mid T_i^{-1}(b_1)T_i^{-1}(b_2'))(E_i^m \mid F_i^n) \\
&= (T_i^{-1}(a_1) \mid T_i^{-1}(b_1))(T_i^{-1}(a_2') \mid T_i^{-1}(b_2'))(E_i^m \mid F_i^n) \\
&= (a_1 \mid b_1)(a_2' \mid b_2')(E_i^m \mid F_i^n) = (a_1 \mid b_1)(a_2 \mid b_2).
\end{aligned}$$

We next prove (9.5) in the case where $B \in \mathfrak{B}^{\infty}$. By Proposition 7.2(2), we have $A(B) = U_{>}(B)$ and $A^{-}(B) = U_{>}^{-}(B)$. Then, by Proposition 5.20(2), there exits $y \in W(B)$ such that $x_2 \in A(y)$ and $y_2 \in A^{-}(y)$, and hence $x_1 \in A(y)^c$ and $y_1 \in A^{-}(y)^c$. Thus we get (9.5) in this case. □

Proposition 9.5. (1) *Let w be an arbitrary element of $\overset{\circ}{W}$. If $X_+ \in A(w,+)$, $Y_+ \in A^{-}(w,+)$, $X_0 \in A(w,0)$, $Y_0 \in A^{-}(w,0)$, $X_- \in A(w,-)$, $Y_- \in A^{-}(w,-)$, then we have*

$$(X_+X_0X_- \,|\, Y_+Y_0Y_-) = (X_+ \,|\, Y_+)(X_0 \,|\, Y_0)(X_- \,|\, Y_-). \tag{9.6}$$

(2) *Let $\preceq$ be an arbitrary convex order on $\Delta(w,-)$. Then the following equality holds*:

$$(E^{\mathbf{c}}_{\succ} \,|\, F^{\mathbf{c}'}_{\succ}) = \delta_{\mathbf{c},\mathbf{c}'} \prod_{\alpha\in\Delta(w,-)} (\mathbf{c}(\alpha))_{q_\alpha}!/(q_\alpha^{-1} - q_\alpha)^{\mathbf{c}(\alpha)}, \tag{9.7}$$

where $F^{\mathbf{c}'}_{\succ} := \Omega(E^{\mathbf{c}'}_{\prec})$, $F^{\mathbf{c}(\alpha)}_{\preceq,\alpha} := \Omega(E^{\mathbf{c}(\alpha)}_{\preceq,\alpha})$.

Proof. (1) Since $X_+X_0 \in A(w,-)^c$ and $Y_+Y_0 \in A^{-}(w,-)^c$, by Proposition 9.4, we have

$$(X_+X_0X_- \,|\, Y_+Y_0Y_-) = (X_+X_0 \,|\, Y_+Y_0)(X_- \,|\, Y_-).$$

Since $A(w,0) \subset A(w,+)^c$ and $A^{-}(w,0) \subset A^{-}(w,+)^c$, by Lemma 9.2(2) and Proposition 9.4, we have

$$(X_+X_0 \,|\, Y_+Y_0) = (X_+ \,|\, Y_+)(X_0 \,|\, Y_0).$$

Hence (9.6) is valid.

(2) We assume that $\mathrm{supp}(\mathbf{c}) \cup \mathrm{supp}(\mathbf{c}') = \{\beta_1, \beta_2, \ldots, \beta_m\}$ and $\beta_1 \prec \beta_2 \prec \cdots \prec \beta_m$, and put $I = (*, \beta_1]_{\preceq}$. Then, by Theorem 8.4(8.22)(8.23), we see that

$$E^{\mathbf{c}(\beta_m)}_{\preceq,\beta_m} \cdots E^{\mathbf{c}(\beta_2)}_{\preceq,\beta_2} \in A(I)^c, \qquad E^{\mathbf{c}(\beta_1)}_{\preceq,\beta_1} \in A(I),$$
$$F^{\mathbf{c}'(\beta_m)}_{\preceq,\beta_m} \cdots F^{\mathbf{c}'(\beta_2)}_{\preceq,\beta_2} \in A^{-}(I)^c, \qquad F^{\mathbf{c}'(\beta_1)}_{\preceq,\beta_1} \in A^{-}(I).$$

Hence, by Lemma 9.2(1), Proposition 9.4, and the induction on m, we see that

$$\begin{aligned}
&(E^{\mathbf{c}}_{\succ} \,|\, F^{\mathbf{c}'}_{\succ}) = (E^{\mathbf{c}(\beta_m)}_{\preceq,\beta_m} \cdots E^{\mathbf{c}(\beta_2)}_{\preceq,\beta_2} E^{\mathbf{c}(\beta_1)}_{\preceq,\beta_1} \,|\, F^{\mathbf{c}'(\beta_m)}_{\preceq,\beta_m} \cdots F^{\mathbf{c}'(\beta_2)}_{\preceq,\beta_2} F^{\mathbf{c}'(\beta_1)}_{\preceq,\beta_1}) \\
&= (E^{\mathbf{c}(\beta_m)}_{\preceq,\beta_m} \cdots E^{\mathbf{c}(\beta_2)}_{\preceq,\beta_2} \,|\, F^{\mathbf{c}'(\beta_m)}_{\preceq,\beta_m} \cdots F^{\mathbf{c}'(\beta_2)}_{\preceq,\beta_2})(E^{\mathbf{c}(\beta_1)}_{\preceq,\beta_1} \,|\, F^{\mathbf{c}'(\beta_1)}_{\preceq,\beta_1}) = \prod_{k=1}^{m} (E^{\mathbf{c}(\beta_k)}_{\preceq,\beta_k} \,|\, F^{\mathbf{c}'(\beta_k)}_{\preceq,\beta_k}) \\
&= \prod_{k=1}^{m} \delta_{\mathbf{c}(\beta_k),\mathbf{c}'(\beta_k)} (E^{\mathbf{c}(\beta_k)}_{\preceq,\beta_k} \,|\, F^{\mathbf{c}(\beta_k)}_{\preceq,\beta_k}) = \delta_{\mathbf{c},\mathbf{c}'} \prod_{\alpha\in\Delta(w,-)} (E^{\mathbf{c}(\alpha)}_{\preceq,\alpha} \,|\, F^{\mathbf{c}(\alpha)}_{\preceq,\alpha}).
\end{aligned}$$

The (9.7) follows from Lemma 9.2(3) and Proposition 9.3. □

Thanks to Proposition 9.5, to complete the computation of values of q-the Killing form, it suffices to compute the values on $A(w,0) \times A^{-}(w,0)$. For the completion of the task, we refer to the following Damiani's work concerning detailed computation of the values of the q-Killing form on the subalgebras generated by the imaginary root vectors.

Proposition 9.6 ([6])**.** (1) *For each $n \in \mathbb{N}$ and $i, j \in \mathring{\mathbf{I}}$ with $i < j$, there is a solution $\{A_{il}^{(n)} \in \mathbb{Q}(q) \,|\, i \le l \in \mathring{\mathbf{I}}\}$ of the following system of linear equations*:

$$\sum_{i \le l} A_{il}^{(n)} (\mathrm{sgn}(\mathrm{A}_{lj}))^n [n\mathrm{A}_{lj}]_{q_l}/n = 0 \tag{9.8}$$

under the condition $A_{ii}^{(n)} \neq 0$.

(2) *For each $(i, n) \in \mathring{\mathbf{I}} \times \mathbb{N}$, one set*

$$\tilde{I}_{i,n} := \sum_{i \le l} A_{il}^{(n)} I_{l,n}, \qquad \tilde{J}_{i,n} := \Omega(\tilde{I}_{i,n}). \tag{9.9}$$

Then the elements $\{\tilde{I}_{i,n} \,|\, n \in \mathbb{N},\, i \in \mathring{\mathbf{I}}\}$ satisfy the following conditions (i)(ii):

(i) *for each $n \in \mathbb{N}$, the sets $\{\tilde{I}_{i,n} \,|\, i \in \mathring{\mathbf{I}}\}$ and $\{I_{i,n} \,|\, i \in \mathring{\mathbf{I}}\}$, respectively, are bases of the same $\mathbb{Q}(q)$-vector subspace of U^+;*

(ii) *for each pair $(\mathbf{c}, \mathbf{c}')$ of finitely supported $\mathbb{Z}_+$-valued functions on $\mathring{\mathbf{I}} \times \mathbb{N}$, the following equality holds*:

$$\Big(\prod_{(i,n) \in \mathring{\mathbf{I}} \times \mathbb{N}} \tilde{I}_{i,n}^{\mathbf{c}(i,n)} \,\Big|\, \prod_{(i,n) \in \mathring{\mathbf{I}} \times \mathbb{N}} \tilde{J}_{i,n}^{\mathbf{c}'(i,n)}\Big) = \delta_{\mathbf{c},\mathbf{c}'} \prod_{(i,n) \in \mathring{\mathbf{I}} \times \mathbb{N}} (\mathbf{c}(i,n))!(\tilde{I}_{i,n} \,|\, \tilde{J}_{i,n})^{\mathbf{c}(i,n)}, \tag{9.10}$$

where the value of $(\tilde{I}_{i,n} \,|\, \tilde{J}_{i,n})$ is given by

$$(\tilde{I}_{i,n} \,|\, \tilde{J}_{i,n}) = A_{ii}^{(n)} \sum_{i \le j} A_{ij}^{(n)} (\mathrm{sgn}(\mathrm{A}_{ji}))^n \frac{[n\mathrm{A}_{ji}]_{q_j}}{n(q_i^{-1} - q_i)}. \tag{9.11}$$

Remark 9.7. *For each $n \in \mathbb{N}$ and $i, j \in \mathring{\mathbf{I}}$ with $i < j$, a solution $\{A_{il}^{(n)} \in \mathbb{Q}(q) \,|\, i \le l \in \mathring{\mathbf{I}}\}$ of the system of the linear equations (9.8) is given in Proposition 7.4.3 in* [6]*.*

Theorem 9.8. *Let $\preceq$ be an arbitrary convex order on Δ_+, and $w \in \mathring{W}$ the unique element such that*

$$\Delta(w, -) \prec \Delta_+^{im} \prec \Delta(w, +). \tag{9.12}$$

We define $\preceq_-$, $\preceq_0$, and $\preceq_+$ to be the restriction of $\preceq$ to $\Delta(w, -)$, Δ_+^{im}, and $\Delta(w, +)$, respectively, and define a total order $\tilde{\preceq}_0$ on the following set

$$\tilde{\Delta}_+^{im} := \Delta_+^{im} \times \mathring{\mathbf{I}} = \{\, (n\delta, i) \mid n \in \mathbb{N},\, i \in \mathring{\mathbf{I}} \,\}$$

by setting

$$(n\delta, i) \,\tilde{\prec}_0\, (n'\delta, i') \iff \begin{cases} n\delta \prec_0 n'\delta & \text{if } n \neq n', \\ i < i' & \text{if } n = n'. \end{cases} \tag{9.13}$$

In addition, we define a total order $\tilde{\preceq}$ on the following set

$$\tilde{\Delta}_+ := \Delta_+^{re} \amalg \tilde{\Delta}_+^{im} = \Delta(w, -) \amalg \tilde{\Delta}_+^{im} \amalg \Delta(w, +)$$

by extending $\preceq_-$, $\tilde{\preceq}_0$, and $\preceq_+$ such as

$$\Delta(w, -) \;\tilde{\prec}\; \tilde{\Delta}_+^{im} \;\tilde{\prec}\; \Delta(w, +). \tag{9.14}$$

For each $\eta \in \tilde{\Delta}_+$*, we set*

$$\tilde{E}_\eta = \tilde{E}_{\preceq,\eta} := \begin{cases} E_{\preceq_-,\eta} & \text{if } \eta \in \Delta(w,-), \\ T_w(\tilde{I}_{i,n}) & \text{if } \eta = (n\delta, i) \in \tilde{\Delta}_+^{im}, \\ \Psi(E_{\preceq_+^{op},\eta}) & \text{if } \eta \in \Delta(w,+), \end{cases} \tag{9.15}$$

where $\preceq_+^{op}$ *is the opposite order of* $\preceq_+$*, and set* $\tilde{F}_\eta = \tilde{F}_{\preceq,\eta} := \Omega(\tilde{E}_{\preceq,\eta})$*. Then the sets* $\tilde{E}_{\succsim}(\tilde{\Delta}_+)$ *and* $\tilde{F}_{\succsim}(\tilde{\Delta}_+)$ *are convex bases of* U^+ *and of* U^- *respectively, and satisfy the following equalities*:

$$(\tilde{E}_{\succsim}^{\mathbf{c}} \mid \tilde{F}_{\succsim}^{\mathbf{c}'}) = \delta_{\mathbf{c},\mathbf{c}'} \prod_{\eta \in \tilde{\Delta}_+} (\mathbf{c}(\eta))_{q_\eta}!(\tilde{E}_{\preceq,\eta} \mid \tilde{F}_{\preceq,\eta})^{\mathbf{c}(\eta)} \tag{9.16}$$

$$= \delta_{\mathbf{c},\mathbf{c}'} \prod_{\alpha \in \Delta_+^{re}} (\mathbf{c}(\alpha))_{q_\alpha}!/(q_\alpha^{-1} - q_\alpha)^{\mathbf{c}(\alpha)} \times \prod_{(n\delta,i) \in \tilde{\Delta}_+^{im}} (\mathbf{c}(n\delta,i))!(\tilde{I}_{i,n} \mid \tilde{J}_{i,n})^{\mathbf{c}(n\delta,i)}, \tag{9.17}$$

where the value of $(\tilde{I}_{i,n} \mid \tilde{J}_{i,n})$ *is given by* (9.11)*. Therefore, the convex basis* $\tilde{E}_{\succsim}(\tilde{\Delta}_+)$ *of* U^+ *and the following convex basis*

$$\left\{ \frac{\tilde{F}_{\succsim}^{\mathbf{c}}}{(\tilde{E}_{\succsim}^{\mathbf{c}} \mid \tilde{F}_{\succsim}^{\mathbf{c}})} \mid \mathbf{c}\colon \tilde{\Delta}_+ \to \mathbb{Z}_+ \ s.t.\ \sharp\mathrm{supp}(\mathbf{c}) < \infty \right\}$$

of U^- *form a pair of the dual bases with respect to the q-Killing form* $(\ \mid\ )$.

Proof. By Proposition 6.8, (7.3), and (9.9), we see that the set $\tilde{E}_{\succsim}(\tilde{\Delta}_+^{im})$ is also a basis of the commutative subalgebra $A(w,0)$. So, by the same manner of the proof of Theorem 8.6, it is easy to see that the first assertion is valid. By Proposition 9.6 and Proposition 9.3, we see that

$$(\tilde{E}_{\succsim}^{\mathbf{c}} \mid \tilde{F}_{\succsim}^{\mathbf{c}'}) = \delta_{\mathbf{c},\mathbf{c}'} \prod_{(n\delta,i) \in \tilde{\Delta}_+^{im}} (\mathbf{c}(n\delta,i))!(\tilde{I}_{i,n} \mid \tilde{J}_{i,n})^{\mathbf{c}(n\delta,i)} \tag{9.18}$$

for each pair $(\mathbf{c}, \mathbf{c}')$ satisfying $\mathrm{supp}(\mathbf{c}), \mathrm{supp}(\mathbf{c}') \subset \tilde{\Delta}_+^{im}$. Let $w_\circ$ be the longest element of $\mathring{W}_{\mathbf{J}}$. Then $\Delta(w,+) = \Delta(ww_\circ,-)$. Hence, by Proposition 9.5(2) and Lemma 9.2(2), we see that

$$(\tilde{E}_{\succsim}^{\mathbf{c}} \mid \tilde{F}_{\succsim}^{\mathbf{c}'}) = \delta_{\mathbf{c},\mathbf{c}'} \prod_{\alpha \in \Delta(w,+)} (\mathbf{c}(\alpha))_{q_\alpha}!/(q_\alpha^{-1} - q_\alpha)^{\mathbf{c}(\alpha)}, \tag{9.19}$$

for each pair $(\mathbf{c}, \mathbf{c}')$ satisfying $\mathrm{supp}(\mathbf{c}), \mathrm{supp}(\mathbf{c}') \subset \Delta(w,+)$. Therefore the equalities (9.16) and (9.17) follows from Proposition 9.5(1)(2), (9.18), and (9.19). The last assertion follows from the first assertion and the equality (9.16). □

Corollary 9.9. *We use the notations as in Proposition 9.6 and Theorem 9.8. For each convex order* $\preceq$ *on* Δ_+*, the universal R-matrix* $\mathcal{R}$ *of* $U_q(\mathfrak{g})$ *can be expressed as follows*:

$$\mathcal{R} = \left(\prod_{\alpha \in \Delta(w,+)}^{\succ} \Theta_{\preceq,\alpha} \right) \left(\prod_{\alpha \in \Delta_+^{im}}^{\succ} \Theta_{\preceq,\alpha} \right) \left(\prod_{\alpha \in \Delta(w,-)}^{\succ} \Theta_{\preceq,\alpha} \right) q^{-T}, \tag{9.20}$$

where $T \in \mathfrak{h}^* \otimes \mathfrak{h}^*$ *is the canonical element of the inner product* $(\ |\ )$ *on* $\mathfrak{h}^*$ *and*

$$\Theta_{\preceq,\alpha} := \begin{cases} \exp_{q_\alpha}\left((q_\alpha^{-1} - q_\alpha)E_{\preceq,\alpha} \otimes F_{\preceq,\alpha}\right) & \text{for } \alpha \in \Delta_+^{re}, \\ \exp\left(\sum_{i=1}^r T_w(\tilde{I}_{i,n}) \otimes T_w(\tilde{J}_{i,n})/(\tilde{I}_{i,n} \mid \tilde{J}_{i,n})\right) & \text{for } \alpha = n\delta \in \Delta_+^{im}. \end{cases} \tag{9.21}$$

Here, $\Theta_{\preceq,\alpha'}$ *is located on the left side of* $\Theta_{\preceq,\alpha}$ *in the product above if* $\alpha' \succ \alpha$*, and* $\exp_{q_\alpha}(x) = \sum_{m=0}^{\infty} x^m/(m)_{q_\alpha}!$.

Proof. Let Θ be the canonical element of the restriction of the q-Killing form to $U_q^+ \times U_q^-$. Then it is known that the universal R-matrix $\mathcal{R}$ of $U_q(\mathfrak{g})$ can be expressed as follows (cf. [15]):

$$\mathcal{R} = \Theta \cdot q^{-T}. \tag{9.22}$$

By Theorem 9.8, we see that

$$\begin{aligned} \Theta &= \sum_{\mathbf{c}} \frac{\tilde{E}_{\tilde{\succ}}^{\mathbf{c}} \otimes \tilde{F}_{\tilde{\succ}}^{\mathbf{c}}}{(\tilde{E}_{\tilde{\succ}}^{\mathbf{c}} \mid \tilde{F}_{\tilde{\succ}}^{\mathbf{c}})} = \sum_{\mathbf{c}} \prod_{\eta \in \tilde{\Delta}_+}^{\tilde{\succ}} \frac{\tilde{E}_{\preceq,\eta}^{\mathbf{c}(\eta)} \otimes \tilde{F}_{\preceq,\eta}^{\mathbf{c}(\eta)}}{(\mathbf{c}(\eta))_{q_\eta}!(\tilde{E}_{\preceq,\eta} \mid \tilde{F}_{\preceq,\eta})^{\mathbf{c}(\eta)}} \\ &= \prod_{\eta \in \tilde{\Delta}_+}^{\tilde{\succ}} \sum_{m=0}^{\infty} \frac{1}{(m)_{q_\eta}!} \left(\frac{\tilde{E}_{\preceq,\eta} \otimes \tilde{F}_{\preceq,\eta}}{(\tilde{E}_{\preceq,\eta} \mid \tilde{F}_{\preceq,\eta})} \right)^m = \prod_{\eta \in \tilde{\Delta}_+}^{\tilde{\succ}} \exp_{q_\eta} \left(\frac{\tilde{E}_{\preceq,\eta} \otimes \tilde{F}_{\preceq,\eta}}{(\tilde{E}_{\preceq,\eta} \mid \tilde{F}_{\preceq,\eta})} \right) \\ &= \left(\prod_{\alpha \in \Delta(w,+)}^{\succ} \Theta_{\preceq,\alpha} \right) \left(\prod_{\eta \in \tilde{\Delta}_+^{im}}^{\tilde{\succ}} \Theta_{\preceq,\eta} \right) \left(\prod_{\alpha \in \Delta(w,-)}^{\succ} \Theta_{\preceq,\alpha} \right), \end{aligned} \tag{9.23}$$

where

$$\Theta_{\preceq,\eta} := \exp\left(T_w(\tilde{I}_{i,n}) \otimes T_w(\tilde{J}_{i,n})/(\tilde{I}_{i,n} \mid \tilde{J}_{i,n})\right) \quad \text{for } \eta = (n\delta, i) \in \tilde{\Delta}_+^{im}. \tag{9.24}$$

Since the elements of $\{T_w(\tilde{I}_{i,n}) \otimes T_w(\tilde{J}_{i,n}) \mid (i,n) \in \mathring{\mathbf{I}} \times \mathbb{N}\}$ are commutative with each other, the imaginary part $\left(\prod_{\eta \in \tilde{\Delta}_+^{im}}^{\tilde{\succ}} \Theta_{\preceq,\eta}\right)$ of the product in (9.23) can be written as

$$\prod_{\eta \in \tilde{\Delta}_+^{im}}^{\tilde{\succ}} \Theta_{\preceq,\eta} = \prod_{n\delta \in \Delta_+^{im}}^{\succ} \prod_{i=1}^{r} \Theta_{\preceq,(n\delta,i)} = \prod_{n\delta \in \Delta_+^{im}}^{\succ} \Theta_{\preceq,n\delta}. \tag{9.25}$$

Therefore the equality (9.20) follows from (9.22), (9.23), and (9.25). □

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